Superinjective Simplicial Maps of Complexes of Curves and Injective Homomorphisms of Subgroups of Mapping Class Groups II

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Abstract

Let R be a compact, connected, orientable surface of genus g with p boundary components. Let $\mathcal{C}(R)$ be the complex of curves on R and Mod_R^* be the extended mapping class group of R. Suppose that either $g = 2$ and $p \geq 2$ or $g \geq 3$ and $p \geq 0$. We prove that a simplicial map $\lambda : \mathcal{C}(R) \to \mathcal{C}(R)$ is superinjective if and only if it is induced by a homeomorphism of R. As a corollary, we prove that if K is a finite index subgroup of Mod_R^* and $f : K \to Mod_R^*$ is an injective homomorphism, then f is induced by a homeomorphism of R and f has a unique extension to an automorphism of Mod_R^*. This extends the author's previous results about closed connected orientable surfaces of genus at least 3, to the surface R.

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1 Introduction

Let R be a compact, connected, orientable surface of genus g with p boundary components. The mapping class group, Mod_R, of R is the group of isotopy classes of orientation preserving homeomorphisms of R. The pure mapping class group, $PMod_R$, is the subgroup of Mod_R consisting of isotopy classes of homeomorphisms which preserve each boundary component of R. The extended mapping class group, Mod_R^*, of R is the group of isotopy classes of all (including orientation reversing) homeomorphisms of R.

Assume that either $g = 2$ and $p \geq 2$ or $g \geq 3$ and $p \geq 0$. The main results of the paper are the following two theorems:

Theorem 1.1 *A simplicial map, $\lambda : \mathcal{C}(R) \to \mathcal{C}(R)$, is superinjective if and only if λ is induced by a homeomorphism of R.*

Theorem 1.2 *Let K be a finite index subgroup of Mod_R^* and f be an injective homomorphism, $f : K \to Mod_R^*$. Then f is induced by a homeomorphism of the surface R (i.e. for some $g \in Mod_R^*$, $f(k) = gkg^{-1}$ for all $k \in K$) and f has a unique extension to an automorphism of Mod_R^*.*

These two theorems were proven for closed, connected, orientable surfaces of genus at least 3 by the author in [3]. They were motivated by the following theorems of Ivanov and the theorem of Ivanov and McCarthy.

Theorem 1.3 (Ivanov) [4] *Let R be a compact, orientable surface possibly with nonempty boundary. Suppose that the genus of R is at least 2. Then, all automorphisms of $\mathcal{C}(R)$ are given by elements of Mod_R^*. More precisely, if R is not a closed surface of genus 2, then $Aut(\mathcal{C}(R)) = Mod_R^*$. If R is a closed surface of genus 2, then $Aut(\mathcal{C}(R)) = Mod_R^*/Center(Mod_R^*)$.*

Theorem 1.4 (Ivanov) [4] *Let R be a compact, orientable surface possibly with nonempty boundary. Suppose that the genus of R is at least 2 and R is not a closed surface of genus 2. Let Γ_1, Γ_2 be finite index subgroups of Mod_R^*. Then, all isomorphisms $\Gamma_1 \to \Gamma_2$ have the form $x \to gxg^{-1}, g \in Mod_R^*$.*

Theorem 1.5 (Ivanov, McCarthy) [6] *Let R and R' be compact, connected, orientable surfaces. Suppose that the genus of R is at least 2, R' is not a closed surface of genus 2, and the maxima of ranks of abelian subgroups of Mod_R and $Mod_{R'}$ differ by at most one. If $h : Mod_R \to Mod_{R'}$ is an injective homomorphism, then h is induced by a homeomorphism $H : S \to R'$, (i.e. $h([G]) = [HGH^{-1}]$, for every orientation preserving homeomorphism $G : R \to R$).*

For the surfaces that we consider in this paper, Theorem 1.1 generalizes Ivanov's Theorem 1.3, Theorem 1.2 generalizes Ivanov's Theorem 1.4 and Ivanov and McCarthy's Theorem 1.5 (in the case when the surfaces are the same).

In section 2, we give some properties of superinjective simplicial maps of the complex of curves, $\mathcal{C}(R)$.

In section 3, we prove that a superinjective simplicial map $\lambda : \mathcal{C}(R) \to \mathcal{C}(R)$, induces an injective simplicial map on the complex of arcs, $\mathcal{B}(R)$, and by using this map we prove that λ is induced by a homeomorphism of R.

In section 4, we prove that if K is a finite index subgroup of Mod_R^* and $f : K \to Mod_R^*$ is an injective homomorphism, then f induces a superinjective simplicial map of $\mathcal{C}(R)$, and by using this map we prove that f is induced by a homeomorphism of R.

2 Superinjective Simplicial Maps of Complexes of Curves

Let R be a compact, connected, orientable surface of genus g with p boundary components. We assume that either $g = 2$ and $p \geq 2$ or $g \geq 3$ and $p \geq 0$. Since our main results have already been proven for $g \geq 3$ and $p = 0$ in [3], we will assume that $p > 0$ when $g \geq 3$ throughout the paper.

A *circle* on R is a properly embedded image of an embedding $S^1 \to R$. A circle on R is said to be *nontrivial* (or *essential*) if it does not bound a disk and it is not homotopic to a boundary component of R. Let C be a collection of pairwise disjoint circles on R. The surface obtained from R by cutting along C is denoted by R_C. A nontrivial circle a on R is called *(k,m)-separating* (or a *(k,m) circle*) if the surface R_a is disconnected and one of its components is a genus k surface with m boundary

components, where $1 \leq k \leq g$, $1 \leq m < p$. If R_a is connected, then a is called *nonseparating.* Let $\mathcal{A}$ denote the set of isotopy classes of nontrivial circles on R. The *geometric intersection number* $i(\alpha, \beta)$ of $\alpha, \beta \in \mathcal{A}$ is the minimum number of points of $a \cap b$ where $a \in \alpha$ and $b \in \beta$.

The *complex of curves*, $\mathcal{C}(R)$, on R, introduced by W. Harvey [2], is an abstract simplicial complex with vertex set $\mathcal{A}$ such that a set of n vertices $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ forms an $n-1$ simplex if and only if $\alpha_1, \alpha_2, ..., \alpha_n$ have pairwise disjoint representatives.

Definition: A simplicial map $\lambda : \mathcal{C}(R) \to \mathcal{C}(R)$ is called **superinjective** if the following condition holds: if α, β are two vertices in $\mathcal{C}(R)$ such that $i(\alpha, \beta) \neq 0$, then $i(\lambda(\alpha), \lambda(\beta)) \neq 0$.

In this section, we show some properties of superinjective simplicial maps of $\mathcal{C}(R)$. The proofs of the Lemmas 2.1-2.4 are similar to the proofs of the corresponding lemmas in the closed case which are given in [3]. So, we will only state them here.

Lemma 2.1 *A superinjective simplicial map, $\lambda : \mathcal{C}(R) \to \mathcal{C}(R)$, is injective.*

Lemma 2.2 *Let α, β be two distinct vertices of $\mathcal{C}(R)$, and let $\lambda : \mathcal{C}(R) \to \mathcal{C}(R)$ be a superinjective simplicial map. Then, α and β are connected by an edge in $\mathcal{C}(R)$ if and only if $\lambda(\alpha)$ and $\lambda(\beta)$ are connected by an edge in $\mathcal{C}(R)$.*

Let P be a set of pairwise disjoint circles on R. P is called a *pair of pants decomposition* of R, if R_P is a disjoint union of genus zero surfaces with three boundary components, pairs of pants. A pair of pants of a pants decomposition is the image of one of these connected components under the quotient map $q : R_P \to R$ together with the image of the boundary components of this component. The image of the boundary of this component is called the *boundary of the pair of pants.* A pair of pants is called *embedded* if the restriction of q to the corresponding component of R_P is an embedding. An ordered set $(a_1, ..., a_{3g-3+p})$ is called an *ordered pair of pants decomposition* of R if $\{a_1, ..., a_{3g-3+p}\}$ is a pair of pants decomposition of R.

Lemma 2.3 *Let $\lambda : \mathcal{C}(R) \to \mathcal{C}(R)$ be a superinjective simplicial map. Let P be a pair of pants decomposition of R. Then, λ maps the set of isotopy classes of elements of P to the set of isotopy classes of elements of a pair of pants decomposition, P', of R.*

Let P be a pair of pants decomposition of R. Let a and b be two distinct elements in P. Then, a is called *adjacent* to b w.r.t. P iff there exists a pair of pants in P which has a and b on its boundary.

Remark: Let P be a pair of pants decomposition of R. Let $[P]$ be the set of isotopy classes of elements of P. Let $\alpha, \beta \in [P]$. We say that α is adjacent to β w.r.t. $[P]$ if the representatives of α and β in P are adjacent w.r.t. P. By Lemma 2.3, λ gives a correspondence on the isotopy classes of elements of pair of pants decompositions of

R. $\lambda([P])$ is the set of isotopy classes of elements of a pair of pants decomposition which corresponds to P, under this correspondence.

Lemma 2.4 *Let $\lambda : \mathcal{C}(R) \to \mathcal{C}(R)$ be a superinjective simplicial map. Let P be a pair of pants decomposition of R. Then, λ preserves the adjacency relation for two circles in P, i.e. if $a, b \in P$ are two circles which are adjacent w.r.t. P and $[a] = \alpha, [b] = \beta$, then $\lambda(\alpha), \lambda(\beta)$ are adjacent w.r.t. $\lambda([P])$.*

Let P be a pair of pants decomposition of R. A curve $x \in P$ is called a *4-curve* in P, if there exist four distinct circles in P, which are adjacent to x w.r.t. P.

Lemma 2.5 *Let $\lambda : \mathcal{C}(R) \to \mathcal{C}(R)$ be a superinjective simplicial map. Let a be a $(k, 1)$-separating circle on R, where $2 \leq k \leq g$. Let R_1, R_2 be the subsurfaces of R s.t. R_1 has genus k and has a as its boundary, and $R_2 = R \setminus R_1 \cup a$. Let $a' \in \lambda([a])$. Then a' is a $(k, 1)$-separating circle and there exist subsurfaces R'_1, R'_2 of R s.t. R'_1 has genus k and has a' as its boundary, $R'_2 = R \setminus R'_1 \cup a'$, $\lambda(\mathcal{C}(R_1)) \subseteq \mathcal{C}(R'_1)$ and $\lambda(\mathcal{C}(R_2)) \subseteq \mathcal{C}(R'_2)$.*

Proof. Let a be a $(k, 1)$-separating circle on R and $2 \leq k \leq g$. Let R_1 be the subsurface of R of genus k which has a as its boundary and $R_2 = R \setminus R_1 \cup a$. If $k = 2$, we choose a pair of pants decomposition $P_1 = \{a_1, a_2, a_3, a_4\}$ of R_1 as shown in Figure 1, (i) and then complete $P_1 \cup \{a\}$ to a pair of pants decomposition P of R in any way we like. If $k \geq 3$, we choose a pair of pants decomposition $P_1 = \{a_1, a_2, ..., a_{3k-2}\}$ of R_1 and then complete $P_1 \cup \{a\}$ to a pair of pants decomposition P of R such that each of a_i is a 4-curve in P for $i = 1, 2, ..., 3k - 2$ and a, a_1, a_3 are the boundary components of a pair of pants of P_1. In Figure 1 (ii), we show how to choose P_1 when $k = 4$. In the other cases, when $k = 3$ or $k \geq 5$, a similar pair of pants decomposition of R_1 can be chosen.

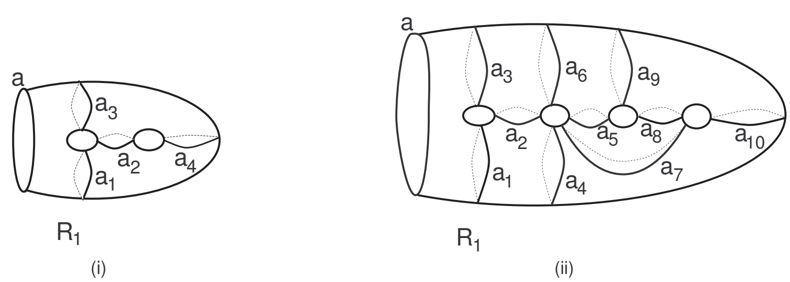


Figure 1: Pants decompositions

Let P' be a pair of pants decomposition of R such that $\lambda([P]) = [P']$. Let a'_i be the representative of $\lambda([a_i])$ which is in P', for $i = 1, ..., 3k - 2$, and a' be the representative of $\lambda([a])$ which is in P'. Let $P'_1 = \{a'_1, a'_2, ..., a'_{3k-2}\}$. By using Lemma 2.4 and following the proof of Lemma 3.5 in [3], we can see that a' is a $(k, 1)$-separating circle, there exist subsurfaces R'_1, R'_2 of R s.t. R'_1 has genus k and has a' as its

boundary, $R'_2 = R \setminus R'_1 \cup a'$ and P'_1 is a pair of pants decomposition of R'_1. Let $P_2 = P \setminus (P_1 \cup a)$ and $P'_2 = P' \setminus (P'_1 \cup a')$. Then P_2, P'_2 are pair of pants decompositions of R_2, R'_2 respectively as P_1, P'_1 are pair of pants decompositions of R_1, R'_1 respectively.

Now, let α be a vertex in $\mathcal{C}(R_1)$. Then, either $\alpha \in [P_1]$ or α has a nonzero geometric intersection with an element of $[P_1]$. In the first case, clearly $\lambda(\alpha) \in \mathcal{C}(R'_1)$ since elements of $[P_1]$ correspond to elements of $[P'_1] \subseteq \mathcal{C}(R'_1)$. In the second case, since λ preserves zero and nonzero geometric intersection (since λ is superinjective) and α has zero geometric intersection with the elements of $[P_2]$ and $[a]$, and nonzero intersection with an element of $[P_1]$, $\lambda(\alpha)$ has zero geometric intersection with elements of $[P'_2]$ and $[a']$, and nonzero intersection with an element of $[P'_1]$. Then, $\lambda(\alpha) \in \mathcal{C}(R'_1)$. Hence, $\lambda(\mathcal{C}(R_1)) \subseteq \mathcal{C}(R'_1)$. The proof of $\lambda(\mathcal{C}(R_2)) \subseteq \mathcal{C}(R'_2)$ is similar. $\square$

Lemma 2.6 *Let $\lambda : \mathcal{C}(R) \to \mathcal{C}(R)$ be a superinjective simplicial map. Then, λ sends the isotopy class of a nonseparating circle to the isotopy class of a nonseparating circle.*

Proof. Let c be a nonseparating circle. Let's choose a separating (2, 1) circle, a on R s.t. c is in genus 2 subsurface, R_1, bounded by a. Then, c can be completed to a pants decomposition $P_1 = \{a_1, a_2, a_3, a_4\}$ on R_1, where $a_4 = c$ as in Figure 1, (i). Then we can complete $P_1 \cup \{a\}$ to a pair of pants decomposition, P, on R in any way we like. Let P' be a pair of pants decomposition of R such that $\lambda([P]) = [P']$. Let a'_i be the representative of $\lambda([a_i])$ which is in P', for $i = 1, ..., 4$, and a' be the representative of $\lambda([a])$ which is in P'. Let $P'_1 = \{a'_1, a'_2, a'_3, a'_4\}$. By Lemma 2.5, a' is a (2,1) separating circle bounding a subsurface, say R'_1, and P'_1 is a pants decomposition on R'_1. Then, by using the adjacency relation between the elements of $P'_1 \cup \{a'\}$ it is easy to see that $a'_4 \in \lambda([c])$ is a nonseparating circle. $\square$

For every 4-curve x in a pants decomposition P, there exist two pairs of pants $A(x)$ and $B(x)$ of P having x as one of their boundary components. Let $C(x) = A(x) \cup B(x)$. The boundary of $C(x)$ consists of four distinct curves, which are adjacent to x w.r.t. P.

Lemma 2.7 *Let $\lambda : \mathcal{C}(R) \to \mathcal{C}(R)$ be a superinjective simplicial map. Then, λ sends the isotopy class of a $(0,3)$-separating circle to the isotopy class of a $(0,3)$-separating circle.*

Proof. Let c be a $(0,3)$-separating circle. A $(0,3)$-separating circle exists if the number of boundary components, p, of R, is at least 2. So, we assume that $p \geq 2$. If $p = 2$, then it is easy to see that a circle is a $(0,3)$-separating circle iff it is a $(g,1)$ separating circle on R. Since c is a $(0,3)$ separating circle, c is a $(g,1)$ separating circle on R. Then, by Lemma 2.5, $\lambda([c])$ has a $(g,1)$ representative which is a $(0,3)$-separating circle on R.

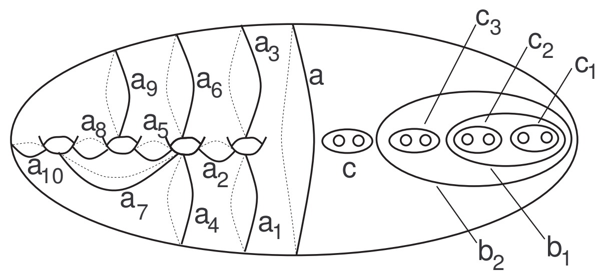


Figure 2: A $(0,3)$ circle, c, in a pants decomposition

Let a be a $(g,1)$-separating circle on R which is disjoint from c. Let R_1 be a subsurface of R of genus g having a as its boundary and $R_2 = R \setminus R_1 \cup a$. Let's choose a pair of pants decomposition P_1 of R_1 as in Lemma 2.5. If the $g = 2$ we choose P_1 as in Figure 1 (i), if $g \geq 3$, we choose P_1 as in Figure 1, (ii).

Assume that $p = 2n$ for some $n \in \mathbb{N}$ and $n \geq 2$.

(i) If $n = 2$, then we choose a pants decomposition P on R s.t. $P = P_1 \cup \{a, c, c_1\}$ where c_1 is a $(0,3)$ curve in P. Let P' be a pair of pants decomposition of R such that $\lambda([P]) = [P']$. Let a', c', c'_1 be the representatives of $\lambda([a]), \lambda([c]), \lambda([c_1])$ in P' respectively. Let $P'_2 = \{c', c'_1\}$. By Lemma 2.5 and Lemma 2.3, there exist subsurfaces R'_1, R'_2 of R of genus g and 0 respectively s.t. R'_1 has a' as its boundary, $R'_2 = R \setminus R'_1 \cup a'$ and P'_2 is a pants decomposition for R'_2. By using Lemma 2.4, we can see that $R'_1 \cup C(a')$ is a genus g surface having c', c'_1 as its boundary components. Since $R'_1 \cup C(a')$ contains P' and P' is a pants decomposition of R, each of c', c'_1 has to be a $(0,3)$ curve.

(ii) If $n > 2$, then we choose a pants decomposition P on R s.t. $P = P_1 \cup \{a, c, b_1, ..., b_{n-2}, c_1, ..., c_{n-1}\}$ where, $b_1, ..., b_{n-2}$ are 4-curves and $c_1, ..., c_{n-1}$ are $(0,3)$ curves in P, as shown in Figure 2 (for $g = 4, p = 8$). In the other cases, a similar pair of pants decomposition of R can be chosen. Let P' be a pair of pants decomposition of R such that $\lambda([P]) = [P']$. Let a'_i be the representative of $\lambda([a_i])$ in P' for $i = 1, ..., 3g-2$, a', c' be the representatives of $\lambda([a]), \lambda([c])$ in P' respectively, b'_i be the representative of $\lambda([b_i])$ in P' for $i = 1, ..., n-2$, c'_i be the representative of $\lambda([c_i])$ in P' for $i = 1, ..., n-1$. Let $P'_1 = \{a'_1, a'_2, ..., a'_{3g-2}\}$, $P'_2 = P' \setminus (P'_1 \cup \{a'\})$.

By Lemma 2.5 and Lemma 2.3, there exist subsurfaces R'_1, R'_2 of R of genus g and 0 respectively s.t. R'_1 has a' as its boundary, $R'_2 = R \setminus R'_1 \cup a'$ and P'_1, P'_2 are pants decompositions for R'_1, R'_2 respectively. By using Lemma 2.4, we can see that $R'_1 \cup C(b'_1) \cup ... \cup C(b'_{n-2})$ is a genus g surface having $c', c'_1, ..., c'_{n-1}$ as its boundary components. Since $R'_1 \cup C(b'_1) \cup ... \cup C(b'_{n-2})$ contains P' and P' is a pants decomposition of R, each of $c', c'_1, ..., c'_n$ has to be a $(0,3)$ curve. Hence, c' is a $(0,3)$ curve.

Assume that $p = 2n + 1$ for some $n \in \mathbb{N}$ and $n \geq 1$.

(i) If $n = 1$, then $P = P_1 \cup \{a, c\}$ is a pants decomposition on R. Let P' be a pair of pants decomposition of R such that $\lambda([P]) = [P']$. Let a', c' be the representatives of $\lambda([a]), \lambda([c])$ in P' respectively. Let $P'_2 = \{c'\}$. By Lemma 2.5 and Lemma 2.3, there exist subsurfaces R'_1, R'_2 of R of genus g and 0 respectively s.t. R'_1 has a' as its boundary, $R'_2 = R \setminus R'_1 \cup a'$ and P'_2 is a pants decomposition for R'_2. Since a' is adjacent to c' in P', there exists a pair of pants Q in R'_2 having a' and c' on its boundary. By using Lemma 2.4, we can see that $R'_1 \cup Q$ is a genus g surface with two boundary components. One of the boundary components is c'. Since $R'_1 \cup Q$ contains P' and P' is a pants decomposition of R, c' has to be a $(0, 3)$ curve.

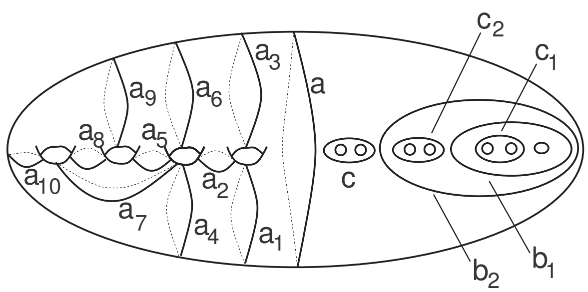


Figure 3: A $(0, 3)$ circle, c, in a pants decomposition

(ii) If $n = 2$, then we choose a pants decomposition P on R s.t. $P = P_1 \cup \{a, c, c_1, b_1\}$ where b_1, c_1 are as in Figure 3. Let P' be a pair of pants decomposition of R such that $\lambda([P]) = [P']$. Let a', c', c'_1, b'_1 be the representatives of $\lambda([a]), \lambda([c]), \lambda([c_1]), \lambda([b_1])$ in P' respectively. Let $P'_2 = \{c', c'_1, b'_1\}$. By Lemma 2.5 and Lemma 2.3, there exist subsurfaces R'_1, R'_2 of R of genus g and 0 respectively s.t. R'_1 has a' as its boundary, $R'_2 = R \setminus R'_1 \cup a'$ and P'_2 is a pants decomposition for R'_2. By using Lemma 2.4, we can see that $R'_1 \cup C(a')$ is a genus g surface having c', b'_1 as its boundary components. Since b_1 is adjacent to c_1 in P, b'_1 is adjacent to c'_1 in P', then there exists a pair of pants Q having b'_1 and c'_1 on its boundary. Then, since $R'_1 \cup C(a') \cup Q$ contains P' and P' is a pants decomposition of R, each of c', c'_1 has to be a $(0, 3)$ curve.

(iii) If $n > 2$, then we choose a pair of pants decomposition P on R s.t. $P = P_1 \cup \{a, c, b_1, ..., b_{n-1}, c_1, ..., c_{n-1}\}$ where P_1 is a pair of pants decomposition as before, $b_2, ..., b_{n-1}$ are 4-curves, b_1 is a 3-curve and $c_1, ..., c_{n-1}$ are $(0, 3)$ curves in P as shown in Figure 3 (for $g = 4, p = 7$). Let P' be a pair of pants decomposition of R such that $\lambda([P]) = [P']$. Let a'_i be the representative of $\lambda([a_i])$ in P' for $i = 1, ..., 3g - 2$, a', c' be the representatives of $\lambda([a]), \lambda([c])$ in P' respectively, b'_i be the representative of $\lambda([b_i])$ in P' for $i = 1, ..., n - 1$, c'_i be the representative of $\lambda([c_i])$ in P' for $i = 1, ..., n - 1$. Let $P'_1 = \{a'_1, a'_2, ..., a'_{3g-2}\}$, $P'_2 = P' \setminus (P'_1 \cup \{a\})$.

Let R'_1, R'_2 be the subsurfaces of R of genus g and 0 respectively s.t. R'_1 has a' as its boundary, $R'_2 = R \setminus R'_1 \cup a'$ and P'_2 is a pants decomposition for R'_2. By using Lemma 2.4, we can see that $R'_1 \cup C(b'_2) \cup ... \cup C(b'_{n-1})$ is a genus g subsurface having $c', c'_2, c'_3, ..., c'_{n-1}, b'_1$ as its boundary components. Notice that b'_1 is adjacent to b'_2, c'_2 in P' and b'_2, c'_2 live in $R'_1 \cup C(b'_2) \cup ... \cup C(b'_{n-1})$. Then, since b_1 is adjacent to c_1 w.r.t. P, there is a pair of pants Q having b'_1 and c'_1 on its boundary. Then, $R'_1 \cup C(b'_2) \cup ... \cup C(b'_{n-1}) \cup Q$ is a genus g subsurface having $c', c'_1, c'_2, ..., c'_{n-1}$ as its boundary components. Since $R'_1 \cup C(b'_2) \cup ... \cup C(b'_{n-1}) \cup Q$ contains P' and P' is a pants decomposition of R, each of $c', c'_1, ..., c'_{n-1}$ has to be a $(0, 3)$ curve. Hence, c' is a $(0, 3)$ curve. $\square$

Let α, β be two distinct vertices in $\mathcal{C}(R)$. We call (α, β) to be a peripheral pair in $\mathcal{C}(R)$ if they have disjoint nonseparating representatives x, y respectively such that x, y and a boundary component of R bound a pair of pants in R.

Lemma 2.8 *Let $\lambda : \mathcal{C}(R) \to \mathcal{C}(R)$ be a superinjective simplicial map and (α, β) be a peripheral pair in $\mathcal{C}(R)$. Then, $(\lambda(\alpha), \lambda(\beta))$ is a peripheral pair in $\mathcal{C}(R)$.*

Proof. Let x, y be disjoint nonseparating representatives of α, β respectively such that x, y and a boundary component of R bound a pair of pants in R.

We will first prove the lemma when $g \geq 3$. Let a be a $(g-1, 1)$-separating circle which is disjoint from x and y. Let R_1 be a subsurface of R of genus $g-1$ having a as its boundary and $R_2 = R \setminus R_1 \cup a$. Let's choose a pair of pants decomposition P_1 of R_1 as in Lemma 2.5 (notice that P_1 was chosen depending on g in Lemma 2.5).

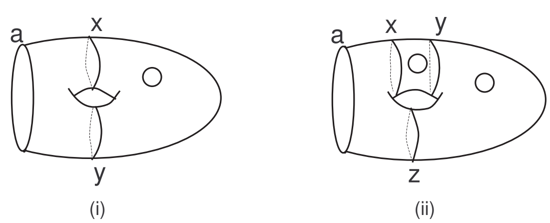


Figure 4: Peripheral pairs

Assume that $p = 1$. Then $P = P_1 \cup \{a, x, y\}$ is a pair of pants decomposition on R (see Figure 4, (i)). Let P' be a pair of pants decomposition of R such that $\lambda([P]) = [P']$. Let a', x', y' be the representatives of $\lambda([a]), \lambda([x]), \lambda([y])$ in P' respectively. Let P'_1 be the set of elements in P' which corresponds to P_1, and let $P'_2 = P' \setminus (P'_1 \cup \{a'\})$. Since $g \geq 3$, by Lemma 2.5, there exist subsurfaces R'_1, R'_2 of R of genus $g-1$ and 1 respectively s.t. R'_1 has a' as its boundary, $R'_2 = R \setminus R'_1 \cup a'$ and $P'_2 = \{x', y'\}$ is a pants decomposition for R'_2. Since a is adjacent to x and y w.r.t. P in R_2, a' has to be adjacent to x' and y' w.r.t. P' in R'_2. So, there exists a pair of pants Q in R'_2 which has a', x', y' on its boundary. Then, $R'_1 \cup Q$

is a genus $g-1$ surface with two boundary components x', y'. Then, since $R'_1 \cup Q$ contains P' and P' is a pants decomposition of R, and R is a genus g surface with 1 boundary component, there has to be a pair of pants having x', y' and the boundary component of R on its boundary. This proves the lemma when $p = 1$.

Assume that $p = 2$. Then $P = P_1 \cup \{a, x, y, z\}$ (see Figure 4, (ii)) is a pair of pants decomposition on R where z is a nonseparating curve, a, x, z are the boundary components of a pair of pants, and $([y], [z])$ is a peripheral pair. Let P' be a pair of pants decomposition of R such that $\lambda([P]) = [P']$. Let a', x', y', z' be the representatives of $\lambda([a]), \lambda([x]), \lambda([y]), \lambda([z])$ in P' respectively. Let P'_1 be the set of elements in P' which corresponds to P_1, and let $P'_2 = P' \setminus (P'_1 \cup \{a'\})$.

By Lemma 2.5, there exist subsurfaces R'_1, R'_2 of R of genus $g-1$ and 1 respectively s.t. R'_1 has a' as its boundary, $R'_2 = R \setminus R'_1 \cup a'$ and $P'_2 = \{x', y', z'\}$ is a pants decomposition for R'_2. Since a is adjacent to x and z w.r.t. P in R_2, a' has to be adjacent to x' and z' w.r.t. P' in R'_2. So, there exists a pair of pants Q in R'_2 which has a', x', z' on its boundary. Then, $R'_1 \cup Q$ is a genus $g-1$ surface with two boundary components x', z'. Then, since x is adjacent to y in R_2, x' is adjacent to y' in R'_2. So, there exists a pair of pants T having x', y' on its boundary. Then, $R'_1 \cup Q \cup T$ is a genus $g-1$ surface with three boundary components. Let w be the boundary component of T which is different from z', y'. Since $R'_1 \cup Q \cup T$ contains P' and P' is a pants decomposition of R, and z' and y' are nonseparating circles by Lemma 2.6, w is a boundary component of R. This proves the lemma for $p = 2$.

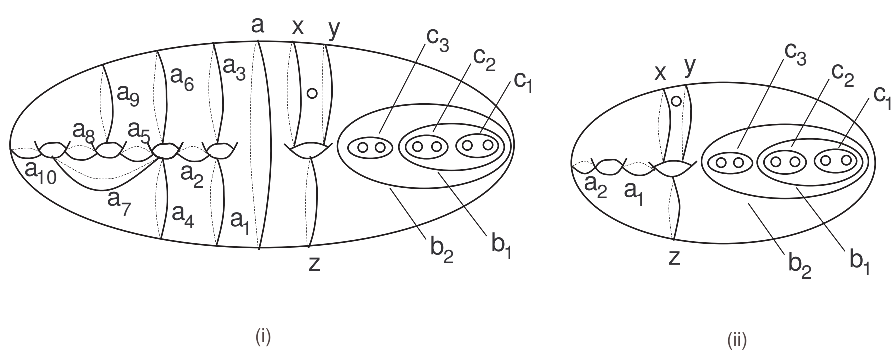


Figure 5: Peripheral pairs pants decompositions

Assume that $p = 2n + 1$ and $n \geq 1$.

(i) If $n = 1$, we choose a pants decomposition P on R s.t. $P = P_1 \cup \{a, x, y, z, c_1\}$ where z is a 4-curve, c_1 is a $(0, 3)$ curve in P, a, x, z bound a pair of pants in P and z, y, c_1 bound a pair of pants in P. Let P' be a pair of pants decomposition of R such that $\lambda([P]) = [P']$. Let a', x', y', z', c'_1 be the representatives of $\lambda([a]), \lambda([x]), \lambda([y]), \lambda([z]), \lambda([c_1])$ in P' respectively. Let P'_1 be the set of elements

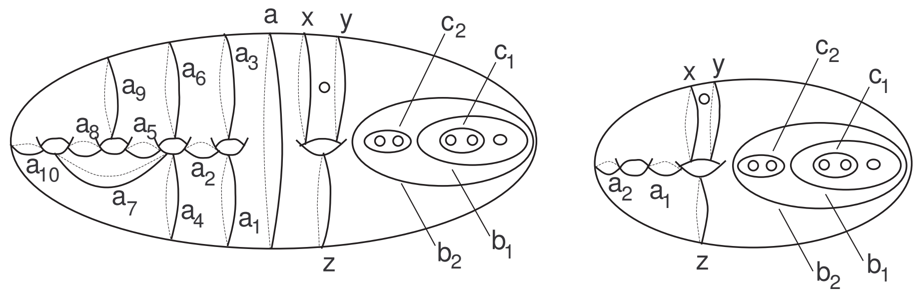
in P' which corresponds to P_1, and let $P'_2 = \{x', y', z', c'_1\}$.

By Lemma 2.5, there exist subsurfaces R'_1, R'_2 of R of genus $g-1$ and 1 respectively s.t. R'_1 has a' as its boundary, $R'_2 = R \setminus R'_1 \cup a'$ and $P'_2 = \{x', y', z', c'_1\}$ is a pants decomposition for R'_2. Since a is adjacent to x and z w.r.t. P in R_2, a' has to be adjacent to x' and z' w.r.t. P' in R'_2. So, there exists a pair of pants Q in R'_2 which has a', x', z' on its boundary. Then, $R'_1 \cup Q$ is a genus $g-1$ surface with two boundary components x', z'. Then, since z is adjacent to y and c_1 w.r.t. P, z' is adjacent to y' and c'_1 w.r.t. P' in R'_2. So, there exists a pair of pants T in R'_2 which has z', y', c'_1 on its boundary. Then, $R'_1 \cup Q \cup T$ is a genus $g-1$ surface with three boundary components x', y', c'_1. Since c_1 is a $(0,3)$ curve in P, c'_1 is a $(0,3)$ curve by Lemma 2.7. Then, since $R'_1 \cup Q \cup T$ contains P' and P' is a pants decomposition of R, there has to be a pair of pants containing x', y' and a boundary component of R as its boundary components.

(ii) If $n > 1$, we choose a pair of pants decomposition P on R such that $P = P_1 \cup \{a, x, y, z, b_1, ..., b_{n-1}, c_1, ..., c_n\}$ where $z, b_1, ..., b_{n-1}$ are 4-curves and $c_1, ..., c_n$ are $(0,3)$ curves in P and a, x, z bound a pair of pants in P. In Figure 5 (i), we show how to choose P when $g = 5, p = 7$. In the other cases, a similar pair of pants decomposition of R can be chosen. Let P' be a pair of pants decomposition of R such that $\lambda([P]) = [P']$. Let a', x', y', z' be the representatives of $\lambda([a]), \lambda([x]), \lambda([y]), \lambda([z])$ in P' respectively, b'_i be the representative of $\lambda([b_i])$ in P' for $i = 1, ..., n-1$ and c'_i be the representative of $\lambda([c_i])$ in P' for $i = 1, ..., n$. Let P'_1 be the set of elements in P' which corresponds to P_1, and let $P'_2 = P' \setminus (P'_1 \cup \{a'\})$.

By Lemma 2.5 and Lemma 2.3, there exist subsurfaces R'_1, R'_2 of R of genus $g-1$ and 1 respectively s.t. R'_1 has a' as its boundary, $R'_2 = R \setminus R'_1 \cup a'$ and P'_2 is a pants decomposition for R'_2. We can see that $R'_1 \cup C(z') \cup C(b'_1) \cup ... \cup C(b'_{n-1})$ is a genus $g-1$ surface having $x', y', c'_1, ..., c'_n$ as its boundary components. Since $c_1, ..., c_n$ are $(0,3)$ curves in P, $c'_1, ..., c'_n$ are $(0,3)$ curves by Lemma 2.7. Then, since $R'_1 \cup C(z') \cup C(b'_1) \cup ... \cup C(b'_{n-1})$ contains P' and P' is a pants decomposition of R, there has to be a pair of pants containing x' and y' and a boundary component of R as its boundary components.

Assume that $p = 2n$, and $n \geq 2$. Let's choose a pants decomposition P on R s.t. $P = P_1 \cup \{a, x, y, z, b_1, ..., b_{n-1}, c_1, ..., c_{n-1}\}$ where, $z, b_2, ..., b_{n-1}$ are 4-curves, b_1 is a 3-curve and $c_1, ..., c_n$ are $(0,3)$ curves in P. In Figure 6 (i), we show how to choose P when $g = 5, p = 6$. Let P' be a pair of pants decomposition of R such that $\lambda([P]) = [P']$. Let a', x', y', z' be the representatives of $\lambda([a]), \lambda([x]), \lambda([y]), \lambda([z])$ in P' respectively, b'_i be the representative of $\lambda([b_i])$ in P' for $i = 1, ..., n-1$, c'_i be the representative of $\lambda([c_i])$ in P' for $i = 1, ..., n-1$. Let P'_1 be the set of elements in P' which corresponds to P_1, and let $P'_2 = P' \setminus (P'_1 \cup \{a'\})$. By Lemma 2.5 and Lemma 2.3, there exist subsurfaces R'_1, R'_2 of R of genus $g-1$ and 1 respectively s.t. R'_1 has a' as its boundary, $R'_2 = R \setminus R'_1 \cup a'$ and P'_2 is a pants decomposition for R'_2.



(i) (ii)

Figure 6: Peripheral pairs in pants decompositions

(i) If $n = 2$, $V = R'_1 \cup C(z')$ is a genus $g-1$ surface having x', y', b'_1 as its boundary components. Since b_1 is adjacent to c_1 w.r.t. P, b'_1 is adjacent to c'_1 w.r.t. P', so there exists a pair of pants Y in $R \backslash V$ s.t. Y contains b'_1, c'_1 on its boundary. Since c_1 is a $(0,3)$ curve in P, c'_1 is a $(0,3)$ curve by Lemma 2.7. Then, since $V \cup Y$ contains P' and P' is a pants decomposition of R, and x' and y' are nonseparating circles, there has to be a pair of pants containing x' and y' and a boundary component of R as its boundary components.

(ii) If $n > 2$, we can see that $W = R'_1 \cup C(z') \cup C(b'_2) \cup ... \cup C(b'_{n-1})$ is a genus $g-1$ surface having $x', y', c'_2, ..., c'_{n-1}, b'_1$ as its boundary components. Since b_1 is adjacent to c_1 w.r.t. P, b'_1 is adjacent to c'_1 w.r.t. P', so there exists a pair of pants Y in $R \backslash W$ s.t. Y contains b'_1, c'_1 on its boundary. Since $c_1, ..., c_{n-1}$ are $(0,3)$ curves in P, $c'_1, ..., c'_{n-1}$ are $(0,3)$ curves by Lemma 2.7. Then, since $W \cup Y$ contains P' and P' is a pants decomposition of R, and x' and y' are nonseparating circles, there has to be a pair of pants containing x' and y' and a boundary component of R as its boundary components. This proves the lemma when $g \geq 3$.

When $g = 2$ the proof is similar. We have $p \geq 2$ in this case. Instead of using a separating curve a, we use pair of pants decompositions as given (for special cases) in Figure 5 (ii) and Figure 6 (ii). By the proof of Lemma 2.5, there are pairwise disjoint representatives a'_1, a'_2, z', x' of $\lambda([a_1]), \lambda([a_2]), \lambda([z]), \lambda([x])$ and a subsurface R'_1 of genus 1 with two boundary components x' and z' such that x', a'_1, a'_2 bound a pair of pants in R'_1, and z', a'_1, a'_2 bound a pair of pants in R'_1. Then, we follow the proof of the case when $g \geq 3$. $\square$

Let M be a sphere with k holes and $k \geq 5$. A circle a on M is called an *n-circle* if a bounds a disk with n holes on M where $n \geq 2$. If a is a 2-circle on M, then there exists up to isotopy a unique nontrivial embedded arc a' on the two-holed disk component of M_a joining the two holes in this disk. If a and b are two 2-circles on M such that the corresponding arcs a', b' can be chosen to meet exactly at

one common end point, and $\alpha = [a], \beta = [b]$, then $\{\alpha, \beta\}$ is called a *simple pair*. A *pentagon* in $\mathcal{C}(M)$ is an ordered 5-tuple $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$, defined up to cyclic permutations, of vertices of $\mathcal{C}(M)$ such that $i(\alpha_j, \alpha_{j+1}) = 0$ for $j = 1, 2, ..., 5$ and $i(\alpha_j, \alpha_k) \neq 0$ otherwise, where $\alpha_6 = \alpha_1$. A vertex in $\mathcal{C}(M)$ is called an *n-vertex* if it has a representative which is an n-circle on M. Let M' be the interior of M. There is a natural isomorphism $\chi : \mathcal{C}(M') \rightarrow \mathcal{C}(M)$ which respects the above notions and the corresponding notions in [7]. Using this isomorphism, we can restate a theorem of Korkmaz as follows:

Theorem 2.9 (Korkmaz) [7] *Let M be a sphere with n holes and $n \geq 5$. Let α, β be two 2-vertices of $\mathcal{C}(\mathcal{M})$. Then $\{\alpha, \beta\}$ is a simple pair iff there exist vertices $\gamma_1, \gamma_2, ..., \gamma_{n-2}$ of $\mathcal{C}(M)$ satisfying the following conditions:*

(i) $(\gamma_1, \gamma_2, \alpha, \gamma_3, \beta)$ is a pentagon in $\mathcal{C}(M)$,

(ii) γ_1 and γ_{n-2} are 2-vertices, γ_2 is a 3-vertex and γ_k and γ_{n-k} are k-vertices for $3 \leq k \leq \frac{n}{2}$,

(iii) $\{\alpha, \gamma_3, \gamma_4, \gamma_5, ..., \gamma_{n-2}\}$, $\{\alpha, \gamma_2, \gamma_4, \gamma_5, ..., \gamma_{n-2}\}$, $\{\beta, \gamma_3, \gamma_4, \gamma_5, ..., \gamma_{n-2}\}$, and $\{\gamma_1, \gamma_2, \gamma_4, \gamma_5, ..., \gamma_{n-2}\}$ are codimension-zero simplices.

Lemma 2.10 *Let $\lambda : \mathcal{C}(R) \rightarrow \mathcal{C}(R)$ be a superinjective simplicial map. Then, λ sends the isotopy class of a (k, m)-separating circle to the isotopy class of a (k, m)-separating circle, where $1 \leq k \leq g$, $1 \leq m < p$.*

Proof. Let $\alpha = [a]$ where a is a (k, m)-separating circle where $1 \leq k \leq g$, $1 \leq m < p$.

Case 1: Assume the genus of R is at least 3. Then a separates a subsurface of genus at least 2. So, it is enough to consider the cases when $k \geq 2$. If $m = 1$, then the lemma follows by Lemma 2.5. Assume that $m \geq 2$. Let R_1 be a subsurface of R of genus k with m boundary components which has a as one of its boundary components. Let's choose a pair of pants decomposition $P_1 = \{a_1, a_2, ...a_{3k-3}, b_1, ..., b_m\}$ of R_1 where $a_1, .., a_{3k-3}$ are 4-curves and (b_i, b_{i+1}) is a peripheral pair for $i = 1, ..., m-1$ as shown in Figure 7 (i) (for $k = 3, m = 5$). Then, we complete $P_1 \cup \{a\}$ to a pair of pants decomposition P of R in any way we like. By Lemma 2.3, we can choose a pair of pants decomposition, P', of R such that $\lambda([P]) = [P']$.

Let a'_i be the representative of $\lambda([a_i])$ which is in P' for $i = 1, .., 3k-3$ and a' be the representative of $\lambda([a])$ which is in P'. Let b'_i be the representative of $\lambda([b_i])$ which is in P' for $i = 1, .., m$. Since a_i is a 4-curve in P, for $i = 1, .., 3k-3$, by Lemma 2.4 and Lemma 2.1, a'_i is a 4-curve in P', for $i = 1, .., 3k-3$. Since (b_i, b_{i+1}) is a peripheral pair, (b'_i, b'_{i+1}) is a peripheral pair for $i = 1, ..., m-1$ by Lemma 2.8. Then, there exist distinct pair of pants Q_i which has b_i, b_{i+1} and a boundary component of R on its boundary for $i = 1, ..., m-1$. Then, it is easy to see that $C(a'_1) \cup ...C(a'_{3k-3}) \cup Q_1 \cup ... \cup Q_{m-1}$ is a genus k subsurface with m boundary components having a' on the boundary. All the other boundary components of this subsurface are boundary components of R. Hence, a' is a (k, m) separating curve.

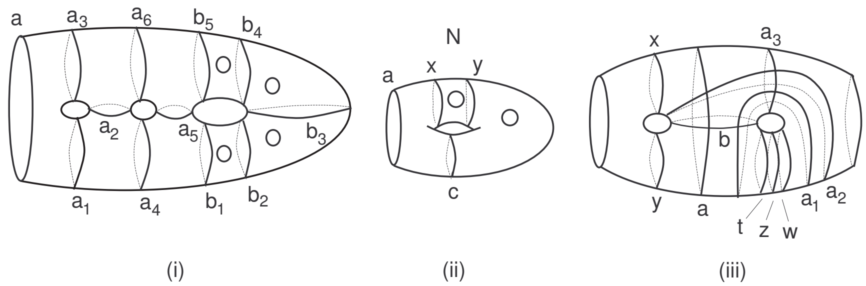


Figure 7: (i) A $(3,5)$ circle a, (ii) A $(1,3)$ circle a, (iii) A $(1,2)$ circle a

Case 2: Assume the genus of R is 2 and the number of boundary components is at least 2. If a separates a subsurface of genus 2, then the proof is similar to the proof of case 1. Assume that a separates a subsurface of genus 1. Then, since the number of boundary components is at least 2, a separates a subsurface of genus 1 with at least 2 boundary components. We will consider the following two cases:

(i) a separates a subsurface N of genus 1 with at least 3 boundary components. We will give the proof when a is a $(1,3)$ circle. The proofs of the remaining cases are similar. Let N and x, y, c be as shown in Figure 7 (ii). We complete $\{a, x, y, c\}$ to a pair of pants decomposition P of R. Let P' be a pair of pants decomposition of R such that $\lambda([P]) = [P']$. Let a', x', y', c' be the representatives of $\lambda([a]), \lambda([x]), \lambda([y]), \lambda([c])$ in P' respectively. Since (x, y) and (c, y) are peripheral pairs, (x', y') and (c', y') are peripheral pairs by Lemma 2.8. Then, there exist distinct pair of pants Q_1, Q_2 such that Q_1 has x', y' and a boundary component of R on its boundary, and Q_2 has c', y' and a boundary component of R on its boundary. There exists also a pair of pants Q_3 which has x', a', c' on its boundary since x' is adjacent to a' and c' w.r.t. P'. Then, it is easy to see that $Q_1 \cup Q_2 \cup Q_3$ is a genus 1 subsurface with 3 boundary components having a' as one of its boundary components and all the other boundary components of this subsurface are boundary components of R. Hence, a' is a $(1,3)$ separating curve.

(ii) Each connected component of R_a is a genus 1 surface with 2 boundary components. Let's choose a pair of pants decomposition $P = \{x, y, z, a_3, b\}$ on R, where the curves are as in Figure 7 (iii). Note that b and a have geometric intersection 2, algebraic 0. Let P' be a pair of pants decomposition of R such that $\lambda([P]) = [P']$. Let x', y', z', a'_3, b' be the representatives of $\lambda([x]), \lambda([y]), \lambda([z]), \lambda([a_3]), \lambda([b])$ in P' respectively. Since (x, y) and (a_3, z) are peripheral pairs, (x', y') and (a'_3, z') are peripheral pairs by Lemma 2.8. Then, there exist distinct pair of pants Q'_1, Q'_2 such that Q'_1 has x', y' and a boundary component of R on its boundary, and Q'_2 has a'_3, z' and a boundary component of R on its boundary. Then, since x' is adjacent to a'_3 and b' w.r.t. P', and y' is adjacent to z' and b' w.r.t. P', there exist also distinct

pair of pants Q'_3 and Q'_4 having x', a'_3, b' and y', z', b' on their boundary respectively. Then, it is easy to see that $R = Q'_1 \cup Q'_2 \cup Q'_3 \cup Q'_4$ and there exists a homeomorphism $\chi : (R, x, y, b, z, a_3) \to (R, x', y', b', z', a'_3)$, i.e. P and P' are topologically equivalent.

Let a_1, a_2, t, w be as shown in Figure 7 (iii). Let Q_2 be the pair of pants in P, with boundary components z, a_3 and a boundary component of R. Let M be the subsurface of R bounded by x, y, t, a_3. M is a sphere with four holes. Let $\tilde{M}$ be the subsurface of R bounded by x, y, t, w and the boundary component of R which is on the boundary of Q_2. $\tilde{M}$ is a sphere with five holes.

Let $t' = \chi(t)$ and $w' = \chi(w)$. Let M' be the subsurface of R bounded by x', y', t', a'_3. M' is a sphere with four holes. Let $\tilde{M}'$ be the subsurface of R bounded by x', y', t', w' and the boundary component of R which is on the boundary of Q'_2. $\tilde{M}'$ is a sphere with five holes.

Since x, y, t, w are essential circles in R, the essential circles on $\tilde{M}$ are essential in R. Similarly, since x', y', t', w' are essential circles in R, the essential circles on $\tilde{M}'$ are essential in R. Furthermore, we can identify $\mathcal{C}(\tilde{M})$ and $\mathcal{C}(\tilde{M}')$ with two subcomplexes of $\mathcal{C}(R)$ in such a way that the isotopy class of an essential circle in $\tilde{M}$ or in $\tilde{M}'$ is identified with the isotopy class of that circle in R. Now, suppose that α is a vertex in $\mathcal{C}(\tilde{M})$. Then, with this identification, α is a vertex in $\mathcal{C}(R)$ and α has a representative in $\tilde{M}$. Then, $i(\alpha, [x]) = i(\alpha, [t]) = i(\alpha, [w]) = i(\alpha, [y]) = 0$. Then there are two possibilities: (i) $\alpha = [b]$ or $\alpha = [a_3]$, (ii) $i(\alpha, [b]) \neq 0$ or $i(\alpha, [a_3]) \neq 0$. Since λ is injective, $\lambda(\alpha)$ is not equal to any of $[x'], [t'], [w'], [y']$. Since λ is superinjective, $i(\lambda(\alpha), [x']) = i(\lambda(\alpha), [t']) = i(\lambda(\alpha), [w']) = i(\lambda(\alpha), [y']) = 0$. Then, there are two possibilities: (i) $\lambda(\alpha) = [b']$ or $\lambda(\alpha) = [a'_3]$, (ii) $i(\lambda(\alpha), [b']) \neq 0$ or $i(\lambda(\alpha), [a'_3]) \neq 0$. Then, a representative of $\lambda(\alpha)$ can be chosen in $\tilde{M}'$. Hence, λ maps the vertices of $\mathcal{C}(R)$ that have essential representatives in $\tilde{M}$ to the vertices of $\mathcal{C}(R)$ that have essential representatives in $\tilde{M}'$, (i.e. λ maps $\mathcal{C}(\tilde{M}) \subseteq \mathcal{C}(R)$ to $\mathcal{C}(\tilde{M}') \subseteq \mathcal{C}(R)$). Similarly, λ maps $\mathcal{C}(M) \subseteq \mathcal{C}(R)$ to $\mathcal{C}(M') \subseteq \mathcal{C}(R)$.

Let $\gamma_1, \gamma_2, \gamma_3$ be the isotopy classes of a_1, a_2, a_3 in $\tilde{M}$ respectively. It is easy to see that $\{[b], [a]\}$ is a simple pair in $\tilde{M}$, $(\gamma_1, \gamma_2, [b], \gamma_3, [a])$ is a pentagon in $\mathcal{C}(\tilde{M})$, γ_1 and γ_3 are 2-vertices, γ_2 is a 3-vertex, and $\{[b], \gamma_3\}$, $\{[b], \gamma_2\}$, $\{[a], \gamma_3\}$ and $\{\gamma_1, \gamma_2\}$ are codimension-zero simplices of $\mathcal{C}(\tilde{M})$.

Since λ is superinjective and x', y', t', w' are essential circles, we can see that $(\lambda(\gamma_1), \lambda(\gamma_2), \lambda([b]), \lambda(\gamma_3), \lambda([a]))$ is a pentagon in $\mathcal{C}(\tilde{M}')$. By the proof of case 2 (i), we can see that $\lambda(\gamma_1)$ is a 2-vertex in $\mathcal{C}(\tilde{M}')$. Using χ, it is easy to see that $\lambda(\gamma_3)$ is a 2-vertex in $\mathcal{C}(\tilde{M}')$. Since (x, a_2) is a peripheral pair in $\tilde{M}$, by using Lemma 2.8 and the existence of χ we can see that (x', a'_2) is a peripheral pair in $\tilde{M}'$ and so $\lambda(\gamma_2)$ is a 3-vertex in $\mathcal{C}(\tilde{M}')$. Since λ is an injective simplicial map $\{\lambda([b]), \lambda(\gamma_3)\}$, $\{\lambda([b]), \lambda(\gamma_2)\}$, $\{\lambda([a]), \lambda(\gamma_3)\}$ and $\{\lambda(\gamma_1), \lambda(\gamma_2)\}$ are codimension-zero simplices of $\mathcal{C}(\tilde{M}')$. Then, by Theorem 2.9, $\{\lambda([b]), \lambda([a])\}$ is a simple pair in $\tilde{M}'$. Since λ maps $\mathcal{C}(M)$ to $\mathcal{C}(M')$, $\lambda([a])$ has a representative a' in M' such that $i(\lambda([b]), \lambda([a]) = |b' \cap a'|$.

Then, there exists a homeomorphism $\theta : (R, x, y, b, a, a_3, z) \to (R, x', y', b', a', a'_3, z')$. Hence, $\lambda([a])$ has a representative a' which is a $(1, 2)$ circle. This proves the lemma for case 2, (ii). $\square$

Lemma 2.11 *Let $\lambda : \mathcal{C}(R) \to \mathcal{C}(R)$ be a superinjective simplicial map. Let t be a (k, m)-separating circle on R, where $1 \leq k \leq g$, $1 \leq m < p$ separating R into two subsurfaces R_1, R_2. Let $t' \in \lambda([t])$. Then t' is a (k, m)-separating circle, t' separates R into two subsurfaces R'_1, R'_2 such that $\lambda(\mathcal{C}(R_1)) \subseteq \mathcal{C}(R'_1)$ and $\lambda(\mathcal{C}(R_2)) \subseteq \mathcal{C}(R'_2)$.*

Proof. Let t be a (k, m)-separating circle where $1 \leq k \leq g$, $1 \leq m < p$. Let R_1, R_2 be the distinct subsurfaces of R of genus k and $g - k$ respectively which come from the separation by t. Let $t' \in \lambda([t])$. By Lemma 2.10, t' is a (k, m)-separating circle. As we showed in the proof of Lemma 2.10, there is a pair of pants decomposition P_1 of R_1, and $P_1 \cup \{t\}$ can be completed to a pair of pants decomposition P of R such that a set of curves, P'_1, corresponding (via λ) to the curves in P_1, can be chosen such that P'_1 is a pair of pants decomposition of a subsurface that has t' as a boundary component and all the other boundary components of this subsurface are boundary components of R. Let R'_1 be this subsurface. Let $R'_2 = R \setminus R'_1 \cup t'$. A pairwise disjoint representative set, P', of $\lambda([P])$ containing $P'_1 \cup \{t'\}$ can be chosen. Then, by Lemma 2.3, P' is a pair of pants decomposition of R. Let $P_2 = P \setminus (P_1 \cup t)$ and $P'_2 = P' \setminus (P'_1 \cup t')$. Then P_2, P'_2 are pair of pants decompositions of R_2, R'_2 respectively as P_1, P'_1 are pair of pants decompositions of R_1, R'_1 respectively.

Now, let α be a vertex in $\mathcal{C}(R_1)$. Then, either $\alpha \in [P_1]$ or α has a nonzero geometric intersection with an element of $[P_1]$. In the first case, clearly $\lambda(\alpha) \in \mathcal{C}(R'_1)$ since elements of $[P_1]$ correspond to elements of $[P'_1] \subseteq \mathcal{C}(R'_1)$. In the second case, since λ preserves zero and nonzero geometric intersection (since λ is superinjective) and α has zero geometric intersection with the elements of $[P_2]$ and $[t]$, and nonzero intersection with an element of $[P_1]$, $\lambda(\alpha)$ has zero geometric intersection with elements of $[P'_2]$ and $[t']$, and nonzero intersection with an element of $[P'_1]$. Then, $\lambda(\alpha) \in \mathcal{C}(R'_1)$. Hence, $\lambda(\mathcal{C}(R_1)) \subseteq \mathcal{C}(R'_1)$. The proof of $\lambda(\mathcal{C}(R_2)) \subseteq \mathcal{C}(R'_2)$ is similar. $\square$

Lemma 2.12 *Let $\lambda : \mathcal{C}(R) \to \mathcal{C}(R)$ be a superinjective simplicial map. Then λ preserves topological equivalence of ordered pairs of pants decompositions of R, (i.e. for a given ordered pair of pants decomposition $P = (c_1, c_2, ..., c_{3g-3+p})$ of R, and a corresponding ordered pair of pants decomposition $P' = (c'_1, c'_2, ..., c'_{3g-3+p})$ of R, where $[c'_i] = \lambda([c_i])$ $\forall i = 1, 2, ..., 3g-3+p$, there exists a homeomorphism $H : R \to R$ such that $H(c_i) = c'_i$ $\forall i = 1, 2, ..., 3g - 3 + p$).*

Proof. Let P be a pair of pants decomposition of R and A be a nonembedded pair of pants in P. The boundary of A consists of the circles x, y where x is a 1-separating circle on R and y is a nonseparating circle on R. Let R_1 be the subsurface of R of genus $g - 1$ with $p + 1$ boundary components which has x as one of its boundary component and let R_2 be the subsurface of R of genus 1 which is bounded by x. Let P_1 be the set of elements of $P \setminus \{x\}$ which are on R_1 and P_2

be the set of elements of $P \setminus \{x\}$ which are on R_2. Then, P_1, P_2 are pair of pants decompositions of R_1, R_2 respectively. So, $P_2 = \{y\}$ is a pair of pants decomposition of R_2. By Lemma 2.11, there exists a 1-separating circle $x' \in \lambda([x])$ and subsurfaces R'_1, R'_2, of R, of genus $g-1$ and 1 respectively such that $\lambda(\mathcal{C}(R_1)) \subseteq \mathcal{C}(R'_1)$ and $\lambda(\mathcal{C}(R_2)) \subseteq \mathcal{C}(R'_2)$. Since $[P_1] \subseteq \mathcal{C}(R_1)$, we have $\lambda([P_1]) \subseteq \mathcal{C}(R'_1)$. Since $[P_2] \subseteq \mathcal{C}(R_2)$, we have $\lambda([P_2]) \subseteq \mathcal{C}(R'_2)$. Since λ preserves disjointness, we can see that a set, P'_1, of pairwise disjoint representatives of $\lambda([P_1])$ disjoint from x' can be chosen. By counting the number of curves in P'_1, we can see that P'_1 is a pair of pants decomposition of R'_1. Similarly, a set, P'_2, of pairwise disjoint representatives of $\lambda([P_2])$ disjoint from x' can be chosen. By counting the number of curves in P'_2, we can see that P'_2 is a pair of pants decomposition of R'_2. Since P_2 has one element, y, P'_2 has one element. Let $y' \in P'_2$. Since x', y' correspond to x, y respectively and y and y' give pair of pants decompositions on R_2 and R'_2 (which are both nonembedded pairs of pants) and x and x' are the boundaries of R_2 and R'_2, we see that λ "sends" a nonembedded pair of pants to a nonembedded pair of pants.

Let B be an embedded pair of pants of P. Let x, y, z be the boundary components of B. There are three cases to consider:

(i) At least one of x, y or z is a separating circle.
(ii) All of x, y, z are nonseparating circles.
(iii) Exactly one of x, y, z is a boundary component of R and the other two are nonseparating circles.

Case (i): W.L.O.G assume that x is a (k, m)-separating circle for $1 \leq k \leq g$, $1 \leq m < p$. Let R_1, R_2 be the distinct subsurfaces of R of genus k and $g-k$ respectively which comes from separation by x. W.L.O.G. assume that y, z are on R_2. Let $x' \in \lambda([x])$. By Lemma 2.11, x' is a (k, m) circle separating R into subsurfaces, R'_1, R'_2, of genus k and $g-k$ respectively such that $\lambda(\mathcal{C}(R_1)) \subseteq \mathcal{C}(R'_1)$ and $\lambda(\mathcal{C}(R_2)) \subseteq \mathcal{C}(R'_2)$.

(a) If y and z are nontrivial circles, then, since $y \cup z \subseteq R_2$, $\lambda(\{[y],[z]\}) \subseteq \mathcal{C}(R'_2)$. Let $y' \in \lambda([y]), z' \in \lambda([z])$ such that $\{x', y', z'\}$ is pairwise disjoint. Let P' be a set of pairwise disjoint representatives of $\lambda([P])$ which contains x', y', z'. P' is a pair of pants decomposition of R. Then, since x is adjacent to y and z w.r.t. P, x' is adjacent to y' and z' w.r.t. P' by Lemma 2.4. Then, since $x' \cup y' \cup z' \subseteq R'_2$, and x' is a boundary component of R'_2, there is an embedded pair of pants in R'_2 which has x', y', z' on its boundary.

(b) If each of y and z is a boundary component of R, then x is a (0,3)-separating circle. Then, by Lemma 2.7, there exists $x' \in \lambda([x])$ s.t. x' is a (0,3)-separating circle. So, there is an embedded pair of pants which has x' and two boundary components of R on its boundary.

(c) W.L.O.G. assume that y is a boundary component of R, and z is a separat-

ing circle. Then, z is a $(k, m+1)$ circle. By using Lemma 2.11, it is easy to see that there exist $z' \in \lambda([z])$ and an embedded pair of pants which has x', z' and a boundary component of R on its boundary.

Hence, λ "sends" an embedded pair of pants to an embedded pair of pants in case (i).

Case (ii): We can find a nonseparating circle w and a $(2,1)$-separating circle t on R such that $\{x, y, z, w\}$ is pairwise disjoint and x, y, z, w are on a genus 2 subsurface, R_1, that t bounds as shown in Figure 8. Let $P_1 = \{x, y, z, w\}$. P_1 is a pair of pants decomposition of R_1. We can complete $P_1 \cup \{t\}$ to a pants decomposition P of R.

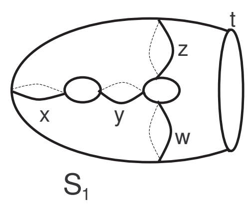


Figure 8: Nonseparating circles, x, y, z, bounding a pair of pants

Let R_2 be the subsurface of R of genus $g-2$ which is not equal to R_1 and that comes from separation by t. By Lemma 2.11, there exist a $(2,1)$-separating circle $t' \in \lambda([t])$ separating R into two subsurfaces, R'_1, R'_2, of genus 2 and $g-2$ respectively such that $\lambda(\mathcal{C}(R_1)) \subseteq \mathcal{C}(R'_1)$ and $\lambda(\mathcal{C}(R_2)) \subseteq \mathcal{C}(R'_2)$. Since $P_1 \subseteq R_1$, $\lambda([P_1]) \subseteq \mathcal{C}(R'_1)$. We can choose a set, P'_1, of pairwise disjoint representatives of $\lambda([P_1])$ on R'_1. Then, $P'_1 \cup \{t'\}$ is a pair of pants decomposition of R'_1. We can choose a pairwise disjoint representative set, P', of $\lambda([P])$ containing P'_1. P' is a pair of pants decomposition of R. Let $x', y', z', w' \in P'_1$ be the representatives of $\lambda([x]), \lambda([y]), \lambda([z]), \lambda([w])$ respectively. Then, since t is adjacent to z and w w.r.t. P, t' is adjacent to z' and w' w.r.t. P' by Lemma 2.4. Then, since $t' \cup z' \cup w' \subseteq R'_1$ and t' is the boundary of R'_1, there is an embedded pair of pants in R'_1 which has t', z', w' on its boundary. Since z is a 4-curve in P, z' is a 4-curve in P'. Since z is adjacent to x, y w.r.t. P, z' is adjacent to x', y' w.r.t. P'. Since z' is on the boundary of a pair of pants which has w', t' on its boundary, and z' is adjacent to x', y', there is a pair of pants having x', y', z' on its boundary. So, λ "sends" an embedded pair of pants bounded by x, y, z to an embedded pair of pants bounded by x', y', z' in this case too.

Case (iii): W.L.O.G assume that z is a boundary component of R and x, y are nonseparating circles. Then, $([x], [y])$ is a peripheral pair. Then, by Lemma 2.8, $(\lambda([x]), \lambda([y]))$ is a peripheral pair. Let x', y' be disjoint representatives of $\lambda([x]), \lambda([y])$ respectively. Then, there exists a pair of pants having x', y' and a boundary component of R on its boundary. Hence, in this case also λ "sends" an embedded pair of pants to an embedded pair of pants.

Now, assume that $P = (c_1, c_2, ..., c_{3g-3+p})$ is an ordered pair of pants decomposition of R. Let $c'_i \in \lambda([c_i])$ such that the elements of $\{c'_1, c'_2, ..., c'_{3g-3+p}\}$ are pairwise disjoint. Then, $P' = (c'_1, c'_2, ..., c'_{3g-3+p})$ is an ordered pair of pants decomposition of R. Let $(B_1, B_2, ..., B_m)$ be an ordered set containing the connected components of R_P. By the arguments given above, there is a corresponding, "image", ordered collection of pairs of pants $(B'_1, B'_2, ..., B'_m)$. Nonembedded pairs of pants correspond to nonembedded pairs of pants and embedded pairs of pants correspond to embedded pairs of pants. Then, the proof of the lemma follows as in the proof of Lemma 3.7 in [3]. □

Remark: Let $\mathcal{E}$ be an ordered set of vertices of $\mathcal{C}(R)$ having a pairwise disjoint representative set E. Then, E can be completed to an ordered pair of pants decomposition, P, of R. We can choose an ordered pairwise disjoint representative set, P', of $\lambda([P])$ by Lemma 2.3. Let E' be the elements of P' which correspond to the elements of E. By Lemma 2.12, P and P' are topologically equivalent as ordered pants decompositions. Hence, the set E and E' are topologically equivalent. So, λ gives a correspondence which preserves topological equivalence on a set which has pairwise disjoint representatives.

By using Lemma 2.10 and following the proof of Lemma 3.9 in [3], we can prove the following lemma. This lemma will be used to see some more properties of superinjective simplicial maps.

Lemma 2.13 *Let $\lambda : \mathcal{C}(R) \to \mathcal{C}(R)$ be a superinjective simplicial map. Let α, β be two vertices of $\mathcal{C}(R)$. If $i(\alpha, \beta) = 1$, then $i(\lambda(\alpha), \lambda(\beta)) = 1$.*

3 Induced Map On Complex Of Arcs

An arc i on R is called *properly embedded* if $\partial i \subseteq \partial R$ and i is transversal to ∂R. i is called *nontrivial* (or *essential*) if i cannot be deformed into ∂R in such a way that the endpoints of i stay in ∂R during the deformation. If a and b are two disjoint arcs connecting a boundary component of R to itself, a and b are called *linked* if their end points alternate on the boundary component. Otherwise, they are called *unlinked*.

The *complex of arcs*, $\mathcal{B}(R)$, on R is an abstract simplicial complex. Its vertices are the isotopy classes of nontrivial properly embedded arcs i in R. A set of vertices forms a simplex if these vertices can be represented by pairwise disjoint arcs.

In this section, we assume that $\lambda : \mathcal{C}(R) \to \mathcal{C}(R)$ is a superinjective simplicial map. Let $\mathcal{V}(R)$ be the set of vertices of $\mathcal{B}(R)$. We prove that λ induces a map $\lambda_* : \mathcal{V}(R) \to \mathcal{V}(R)$ with certain properties. Then we prove that λ_* extends to an injective simplicial map $\lambda_* : \mathcal{B}(R) \to \mathcal{B}(R)$.

The proofs of Lemma 3.1 - 3.5 are similar to the proofs of the corresponding lemmas given in [3]. So, we do not prove these lemmas here. We only state them.

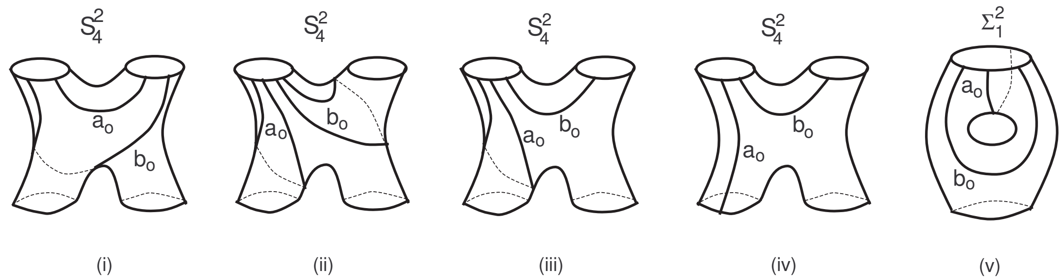


Figure 9: Disjoint arcs and neighborhoods

Lemma 3.1 *Let a and b be two disjoint arcs on R connecting two distinct boundary components, ∂_i, ∂_j, of R. Let N be a regular neighborhood of $a \cup b \cup \partial_1 \cup \partial_2$ in R. Then, $(N, a, b) \cong (S_4^2, a_o, b_o)$ where S_4^2 is a standard sphere with four holes and a_o, b_o are arcs as shown in Figure 9, (i).*

Lemma 3.2 *Let a and b be two disjoint arcs which are unlinked, connecting one boundary component ∂_i of R to itself. Let N be a regular neighborhood of $a \cup b \cup \partial_i$ on R. Then, $(N, a, b) \cong (S_4^2, a_o, b_o)$ where a_o, b_o are the arcs drawn on a standard sphere with four holes, S_4^2, as shown in Figure 9, (ii).*

Lemma 3.3 *Let a and b be two disjoint arcs on R such that a connects one boundary component ∂_i of R to itself for some $k = 1, ..., p$ and b connects the boundary components ∂_i and ∂_j of R, where $i \neq j$. Let N be a regular neighborhood of $a \cup b \cup \partial_i \cup \partial_j$. Then, $(N, a, b) \cong (S_4^2, a_o, b_o)$ where a_o, b_o are the arcs drawn on a standard sphere with four holes, S_4^2, as shown in Figure 9, (iii).*

Lemma 3.4 *Let a and b be two disjoint arcs. Suppose that a connects ∂_i to ∂_j and b connects ∂_i to ∂_k where ∂_i, ∂_j, ∂_k are three distinct boundary components. Let N be a regular neighborhood of $a \cup b \cup \partial_i \cup \partial_j \cup \partial_k$. Then, $(N, a, b) \cong (S_4^2, a_o, b_o)$ where a_o, b_o are the arcs drawn on a standard sphere with four holes, S_4^2, as shown in Figure 9, (iv).*

Lemma 3.5 *Let a and b be two disjoint, linked arcs connecting one boundary component ∂_i of R to itself for $i = 1, ..., p$. Let N be a regular neighborhood of $a \cup b \cup \partial_i$. Then, $(N, a, b) \cong (\Sigma_1^2, a_o, b_o)$ where Σ_1^2 is a standard surface of genus one with two boundary components, and a_o, b_o are as shown in Figure 9, (v).*

By using the following lemmas, we see some more properties of λ.

Lemma 3.6 *Let α and β be two vertices in $\mathcal{C}(R)$ which have representatives with geometric intersection 2 and algebraic intersection 0 on R. Then, $\lambda(\alpha)$ and $\lambda(\beta)$ have representatives with geometric intersection 2 and algebraic intersection 0 on R.*

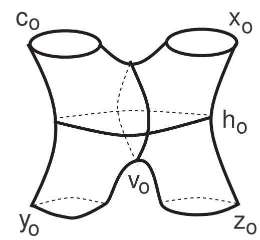


Figure 10: "Horizontal" and "vertical" circles

Proof. Let h, v be representatives of α, β with geometric intersection 2 and algebraic intersection 0 on R. W.L.O.G assume that h and v do not intersect the boundary components of R. Let N be a regular neighborhood of $h \cup v$ on $Int(R)$. Then, N is a sphere with four boundary components. Let c, x, y, z be boundary components of N such that there exists a homeomorphism $\varphi : (N, c, x, y, z, h, v) \to (N_o, c_o, x_o, y_o, z_o, h_o, v_o)$ where N_o is a standard sphere with four holes having c_o, x_o, y_o, z_o on its boundary and h_o, v_o (horizontal, vertical) are two circles as indicated in Figure 10. Since h and v have geometric intersection 2 and algebraic intersection 0 on R, none of c, x, y, z bound a disk on R. If each of c, x, y, z is an essential circle on R, and R is not a surface of genus two with two boundary components, then the proof of the lemma follows from the proof of Lemma 4.6 in [3], substituting Lemma 2.12 above for the corresponding lemma in [3]. If each of c, x, y, z is an essential circle on R, and R is a surface of genus two with two boundary components, then the proof of the lemma follows from the proof of Lemma 2.10. Assume that exactly one of c, x, y, z is not essential. W.L.O.G assume that c is not essential. Then, since N is a regular neighborhood in $Int(R)$, and c does not bound a disk on R, c and a boundary component, say ∂_1, of R bound an annulus, A. Let $M = N \cup A$. Then, M is a regular neighborhood of $h \cup v$.

Let A= $\{x, y, z\}$. Any two elements in A which are isotopic in R bound an annulus on R. Let B be a set consisting of a core from each annulus which is bounded by elements in A, circles in A which are not isotopic to any other circle in A, and v. We can extend B to a pants decomposition P of R. Since either $g = 2$ and $p \geq 2$ or $g \geq 3$ and $p > 0$, there are at least four pairs of pants of P. Note that $\{v\}$ is a pair of pants decomposition of M. Each pair of pants of this pants decomposition of M is contained in exactly one pair of pants in P. It is easy to see that there is a pair of pants Q of P such that interior of Q is disjoint from M, Q has at least one of x, y, z as one of its boundary components and all the boundary components of Q are essential circles in R. We will give the argument for the case where Q has y on

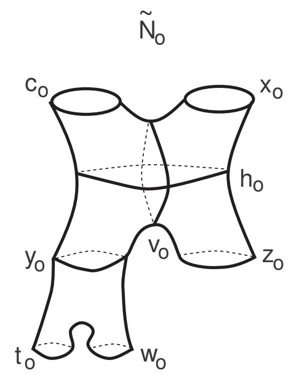


Figure 11: Sphere with five holes

its boundary. The other cases follow by similar arguments.

Let T be a regular neighborhood, in Q, of the boundary components of Q other than y. Let t, w be the boundary components of T which are in the interior of Q. Then, y, t, w bound an embedded pair of pants O in Q. Let $\tilde{M} = M \cup O$. Then, we can extend N_o to $\tilde{N}_o$ and find a homeomorphism $\tilde{\varphi} : (\tilde{M}, \partial_1, x, y, z, h, v, t, w) \rightarrow (\tilde{N}_o, c_o, x_o, y_o, z_o, h_o, v_o, t_o, w_o)$, where $\tilde{N}_o$ is as shown in Figure 11.

Using Lemma 2.12, we can choose pairwise disjoint representatives x', y', z', v', t', w' of $\lambda([x]), \lambda([y]), \lambda([z]), \lambda([v]), \lambda([t]), \lambda([w])$ respectively s.t. there exists a subsurface $\tilde{M}'$ of R and a homeomorphism $\chi : (\tilde{M}', \partial_i, x', y', z', v', t', w') \rightarrow (\tilde{N}_o, c_o, x_o, y_o, z_o, v_o, t_o, w_o)$ for some boundary component ∂_i of R. Since $i([h], [v]) \neq 0$ and λ is superinjective, we have, $i(\lambda([h]), \lambda([v])) \neq 0$. Then, a representative h' of $\lambda([h])$ can be chosen such that h' is transverse to v', h' doesn't intersect any of ∂_i, x', y', z', and $i(\lambda([h]), \lambda([v])) = |h' \cap v'|$. Since $i(\lambda([h]), \lambda([v])) \neq 0$, h' intersects v'. Hence, h' is in the sphere with four holes bounded by ∂_i, x', y', z' in $\tilde{M}'$.

$\tilde{M}$ and $\tilde{M}'$ are spheres with five holes in R. Since x, z, t, w are essential circles in R, the essential circles on $\tilde{M}$ are essential in R. Similarly, since x', z', t', w' are essential circles in R, the essential circles on $\tilde{M}'$ are essential in R. Furthermore, we can identify $\mathcal{C}(\tilde{M})$ and $\mathcal{C}(\tilde{M}')$ with two subcomplexes of $\mathcal{C}(R)$ in such a way that the isotopy class of an essential circle in $\tilde{M}$ or in $\tilde{M}'$ is identified with the isotopy class of that circle in R. Now, suppose that α is a vertex in $\mathcal{C}(\tilde{M})$. Then, with this identification, α is a vertex in $\mathcal{C}(R)$ and α has a representative in $\tilde{M}$. Then, $i(\alpha, [x]) = i(\alpha, [t]) = i(\alpha, [w]) = i(\alpha, [z]) = 0$. Then there are two possibilities: (i) $\alpha = [v]$ or $\alpha = [y]$, (ii) $i(\alpha, [v]) \neq 0$ or $i(\alpha, [y]) \neq 0$. Since λ is injective, $\lambda(\alpha)$ is not equal to any of $[x'], [t'], [w'], [z']$. Since λ is superinjective, $i(\lambda(\alpha), [x']) = i(\lambda(\alpha), [t']) = i(\lambda(\alpha), [w']) = i(\lambda(\alpha), [z']) = 0$. Then, there are two possibilities: (i) $\lambda(\alpha) = [v']$ or $\lambda(\alpha) = [y']$, (ii) $i(\lambda(\alpha), [v']) \neq 0$ or $i(\lambda(\alpha), [y']) \neq 0$. Then, a representative of $\lambda(\alpha)$ can be chosen in $\tilde{M}'$. Hence, λ maps the vertices

of $\mathcal{C}(R)$ that have essential representatives in $\tilde{M}$ to the vertices of $\mathcal{C}(R)$ that have essential representatives in $\tilde{M}'$, (i.e. λ maps $\mathcal{C}(\tilde{M}) \subseteq \mathcal{C}(R)$ to $\mathcal{C}(\tilde{M}') \subseteq \mathcal{C}(R)$). Similarly, λ maps $\mathcal{C}(M) \subseteq \mathcal{C}(R)$ to $\mathcal{C}(M') \subseteq \mathcal{C}(R)$.

It is easy to see that $\{[h], [v]\}$ is a simple pair in $\tilde{M}$. Then, by Theorem 2.9, there exist vertices $\gamma_1, \gamma_2, \gamma_3$ of $\mathcal{C}(\tilde{M})$ such that $(\gamma_1, \gamma_2, [h], \gamma_3, [v])$ is a pentagon in $\mathcal{C}(\tilde{M})$, γ_1 and γ_3 are 2-vertices, γ_2 is a 3-vertex, and $\{[h], \gamma_3\}$, $\{[h], \gamma_2\}$, $\{[v], \gamma_3\}$ and $\{\gamma_1, \gamma_2\}$ are codimension-zero simplices of $\mathcal{C}(\tilde{M})$.

Since λ is superinjective and x', t', w', z' are essential circles, we can see that $(\lambda(\gamma_1), \lambda(\gamma_2), \lambda([h]), \lambda(\gamma_3), \lambda([v]))$ is a pentagon in $\mathcal{C}(\tilde{M}')$. By Lemma 2.12, $\lambda(\gamma_1)$ and $\lambda(\gamma_3)$ are 2-vertices, and $\lambda(\gamma_2)$ is a 3-vertex in $\mathcal{C}(\tilde{M}')$. Since λ is an injective simplicial map $\{\lambda([h]), \lambda(\gamma_3)\}$, $\{\lambda([h]), \lambda(\gamma_2)\}$, $\{\lambda([v]), \lambda(\gamma_3)\}$ and $\{\lambda(\gamma_1), \lambda(\gamma_2)\}$ are codimension-zero simplices of $\mathcal{C}(\tilde{M}')$. Then, by Theorem 2.9, $\{\lambda([h]), \lambda([v])\}$ is a simple pair in $\tilde{N}'$. Since $\lambda([h])$ has a representative, h', in M', such that $i(\lambda([h]), \lambda([v]) = |h' \cap v'|$ and $\{\lambda([h]), \lambda([v])\}$ is a simple pair in $\tilde{M}'$, there exists a homeomorphism $\chi : (M', \partial_i, x', y', z', h', v') \to (N_o, c_o, x_o, y_o, z_o, h_o, v_o)$.

The proof of the lemma in the remaining cases, when M has more than one inessential boundary component is similar to the previous case. □

Lemma 3.7 *Let c, x be curves which are either essential circles on R or some boundary components of R. Let y, z, m, n be essential circles on R such that there exists a subsurface N of R and a homeomorphism $\varphi : (N, c, x, y, z, m, n) \to (N_o, c_o, x_o, y_o, z_o, m_o, n_o)$ where N_o is a standard torus with two boundary components, c_o, x_o, and y_o, z_o, m_o, n_o are circles as shown in Figure 12 (i). Then, there exist c', x', two simple closed curves, and $y' \in \lambda([y]), z' \in \lambda([z]), m' \in \lambda([m]), n' \in \lambda([n]), N' \subseteq R$ and a homeomorphism $\chi : (N', c', x', y', z', m', n') \to (N_o, c_o, x_o, y_o, z_o, m_o, n_o)$.*

Proof. If both c and x are essential circles, then the proof follows from Lemma 4.7 in [3], substituting Lemma 2.12 above for the corresponding lemma in [3].

Since the genus of R is at least 2, both of c and x cannot be boundary components of R. W.L.O.G assume that x is essential and c is a boundary component of R. We can complete $\{x, y, z\}$ to a pair of pants decomposition, P, of R. Since $\{y, z\}$ gives a pair of pants decomposition on N, by Lemma 2.12, there exists a subsurface $N' \subseteq R$ which is homeomorphic to N_o and there are pairwise disjoint representatives x', y', z' of $\lambda([x]), \lambda([y]), \lambda([z])$ respectively and a homeomorphism ϕ such that $(N', \partial_i, x', y', z') \to_\phi (N_o, c_o, x_o, y_o, z_o)$ for some $i \in \{1, ..., k\}$. Then by Lemma 2.13, we have the following:

$i([m], [z]) = 1 \Rightarrow i(\lambda([m]), \lambda([z])) = 1$, $i([n], [z]) = 1 \Rightarrow i(\lambda([n]), \lambda([z])) = 1$, $i([m], [y]) = 1 \Rightarrow i(\lambda([m]), \lambda([y])) = 1$, $i([n], [y]) = 1 \Rightarrow i(\lambda([n]), \lambda([y])) = 1$, $i([m], [x]) = 0 \Rightarrow i(\lambda([m]), \lambda([x])) = 0$, $i([n], [x]) = 0 \Rightarrow i(\lambda([n]), \lambda([x])) = 0$, $i([m], [n]) = 0 \Rightarrow i(\lambda([m]), \lambda([n])) = 0$.

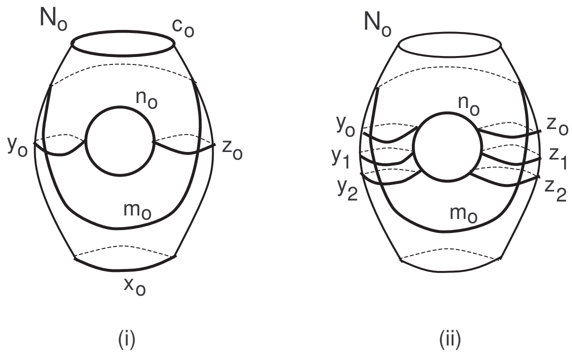


Figure 12: Circles on torus with two boundary components

Let $c' = \partial_i$. There are representatives $m_1 \in \lambda([m]), n' \in \lambda([n])$ such that $|m_1 \cap y'| = |m_1 \cap z'| = 1$, $|m_1 \cap c'| = |m_1 \cap x'| = 0$, $|n' \cap y'| = |n' \cap z'| = 1, |n' \cap c'| = |n' \cap x'| = |m_1 \cap n'| = 0$ with all intersections transverse.

Since ϕ is a homeomorphism, we have $|\phi(m_1) \cap \phi(n')| = 0, |\phi(n') \cap y_o| = 1$, $|\phi(n') \cap z_o| = 1, |\phi(n') \cap c_o| = |\phi(n') \cap x_o| = 0 = |\phi(m_1) \cap c_o| = |\phi(m_1) \cap x_o| = 0$, $|\phi(m_1) \cap y_o| = |\phi(m_1) \cap z_o| = 1$.

Let's choose parallel copies y_1, y_2 of y_o and z_1, z_2 of z_o as shown in Figure 12 (ii) so that each of them has transverse intersection one with $\phi(m_1)$ and $\phi(n')$. Let P_1, P_2 be the pair of pants with boundary components c_o, y_o, z_o, and x_o, y_2, z_2 respectively. Let Q_1, Q_2, R_1, R_2 be the annulus with boundary components $\{y_o, y_1\}, \{y_1, y_2\}$, $\{z_o, z_1\}$, $\{z_1, z_2\}$ respectively. By the classification of isotopy classes of families of properly embedded disjoint arcs in pairs of pants, $\phi(m_1) \cap P_1$, $\phi(m_1) \cap P_2$, $\phi(n') \cap P_1$ and $\phi(n') \cap P_2$ can be isotoped to the arcs $m_o \cap P_1, m_o \cap P_2, n_o \cap P_1, n_o \cap P_2$ respectively. Let $\kappa : P_1 \times I \to P_1$, $\tau : P_2 \times I \to P_2$ be such isotopies. By a tapering argument, we can extend κ and τ and get $\tilde{\kappa} : (P_1 \cup Q_1 \cup R_1) \times I \to (P_1 \cup Q_1 \cup R_1)$ and $\tilde{\tau} : (P_2 \cup Q_2 \cup R_2) \times I \to (P_2 \cup Q_2 \cup R_2)$ so that $\tilde{\kappa}_t$ is id on $y_1 \cup z_1$ and $\tilde{\tau}_t$ is id on $y_1 \cup z_1$ for all $t \in I$. Then, by gluing these extensions we get an isotopy ϑ on $N_o \times I$.

By the classification of isotopy classes of arcs (relative to the boundary) on an annulus, $\vartheta_1(\phi(n')) \cap (R_1 \cup R_2)$ can be isotoped to $t_{z_o}^k(n_o) \cap (R_1 \cup R_2)$ for some $k \in \mathbb{Z}$. Let's call this isotopy μ. Let $\tilde{\mu}$ denote the extension by id to N_o. Similarly, $\vartheta_1(\phi(n')) \cap (Q_1 \cup Q_2)$ can be isotoped to $t_{y_o}^l(n_o) \cap (Q_1 \cup Q_2)$ for some $l \in \mathbb{Z}$. Let's call this isotopy ν. Let $\tilde{\nu}$ denote the extension by id to N_o. Then, "gluing" the two isotopies $\tilde{\mu}$ and $\tilde{\nu}$, we get a new isotopy, ϵ, on N_o. Then we have, $t_{y_o}^{-l}(t_{z_o}^{-k}(\epsilon_1(\vartheta_1(\phi(n'))))) = n_o$. Clearly, $t_{y_o}^{-l} \circ t_{z_o}^{-k} \circ \epsilon_1 \circ \vartheta_1$ fixes c_o, x_o, y_o, z_o. So, we get $t_{y_o}^{-l} \circ t_{z_o}^{-k} \circ \epsilon_1 \circ \vartheta_1 \circ \phi : (N', c', x', y', z', n') \to (N_o, c_o, x_o, y_o, z_o, n_o)$. Let $\chi = t_{y_o}^{-l} \circ t_{z_o}^{-k} \circ \epsilon_1 \circ \vartheta_1 \circ \phi$. Then, because of the intersection information we also have that $\chi(m_1)$ is isotopic to either m_o or $\hat{m}_o$ where $\hat{m}_o$ is the curve that we get from m_o

by reflecting the picture in Figure 12 (i) about the plane of the paper. Let ρ be this reflection. We have $\rho(m_o) = \hat{m}_o$. If $\chi(m_1)$ is isotopic to m_o, we let $m' = \chi^{-1}(m_o)$, and we get $\chi : (N', c', x', y', z', m', n') \to (N_o, c_o, x_o, y_o, z_o, m_o, n_o)$. If $\chi(m_1)$ is isotopic to $\hat{m}_o$, we let $m' = \chi^{-1}(\hat{m}_o)$, and we get $\rho^{-1} \circ \chi : (N', c', x', y', z', m', n') \to (N_o, c_o, x_o, y_o, z_o, m_o, n_o)$. This proves the lemma. $\square$

Let i be an essential properly embedded arc on R. Let A be a boundary component of R which has one end point of i and B be the boundary component of R which has the other end point of i. Let N be a regular neighborhood of $i \cup A \cup B$ in R. By Euler characteristic arguments, N is a pair of pants. The boundary components of N are called *encoding circles of i on R*. The set of isotopy classes of nontrivial encoding circles on R is called the *encoding simplex*, Δ_i, of i (and of $[i]$).

An essential properly embedded arc i on R is called *type 1* if it joins one boundary component ∂_k of R to itself. It is called *type 1.1* if $\partial_k \cup i$ has a regular neighborhood N in R which has only one circle on its boundary which is inessential w.r.t. R. If N has two circles on its boundary which are inessential w.r.t. R, then i is called *type 1.2*. We call i to be *type 2*, if it joins two different boundary components of R to each other. An element $[i] \in \mathcal{V}(R)$ is called *type 1.1 (1.2, 2)* if it has a type 1.1 (1.2, 2) representative. i is called *nonseparating* if its complement in R is connected.

Let $\partial^1, \partial^2, ..., \partial^p$ be the boundary components of R. We prove the following lemmas in order to show that λ induces a map $\lambda_* : \mathcal{V}(R) \to \mathcal{V}(R)$ with certain properties.

Lemma 3.8 *Let $\partial_k \subseteq \partial R$ for some $k \in \{1, ..., p\}$. Then, there exists a unique $\partial^l \in \partial R$ for some $l \in \{1, ..., p\}$ such that if i is a properly embedded essential arc on R connecting ∂_k to itself, then there exists a properly embedded arc j on R connecting ∂^l to itself such that $\lambda(\Delta_i) = \Delta_j$.*

Proof. Assume that there are two boundary components, ∂^r and ∂^t such that each of them satisfies the hypothesis. Let i be a properly embedded, essential, nonseparating type 1 arc connecting ∂_k to itself. Then, there exist properly embedded arcs, j_1, connecting ∂^r to itself, and j_2, connecting ∂^t to itself, such that $\lambda(\Delta_i) = \Delta_{j_1}$ and $\lambda(\Delta_i) = \Delta_{j_2}$. Then, we have $\Delta_{j_1} = \Delta_{j_2}$. Note that a properly embedded essential arc i is type 1.1 iff Δ_i has exactly 2 elements. Otherwise Δ_i has 1 element. Since i is nonseparating type 1, it is type 1.1. Then, since $\lambda(\Delta_i) = \Delta_{j_1}$ and $\lambda(\Delta_i) = \Delta_{j_2}$ and λ is injective, j_1 and j_2 are type 1.1. We can choose a pairwise disjoint representative set $\{a, b\}$ of Δ_{j_1} on R. Since $\Delta_{j_1} = \Delta_{j_2}$, $\{a, b\}$ is a pairwise disjoint representative set for Δ_{j_2} on R. Then, a, b and ∂^r bound a pair of pants, P, on R containing an arc, j_1', isotopic to j_1. Similarly, a, b, ∂^t bound a pair of pants, Q, on R containing an arc, j_2', isotopic to j_2. Let's cut R along a and b. Then, P is the connected component of $R_{a \cup b}$ containing ∂^r and Q is the connected component of $R_{a \cup b}$ containing ∂^t. $P \neq Q$ since ∂^t is not in P and ∂^t is in Q. Then P and Q are distinct connected components meeting along a and b. Hence, R is $P \cup Q$, a torus with two holes which gives a contradiction since the genus of R is at least 2. So,

only one boundary component of R can satisfy the hypothesis.

Since i is nonseparating type 1, Δ_i contains two isotopy classes of nontrivial circles in R. Let P' be a pairwise disjoint representative set of $\lambda([\Delta_i])$. Since the genus of R is at least 2, by the proof of Lemma 2.12, we can see that P' and a boundary component of R bounds a unique pair of pants Q, in R which has only one inessential boundary component. Let $\partial^{l(i)}$ be this inessential boundary component. Let j be an essential properly embedded arc connecting $\partial^{l(i)}$ to itself in Q. Then, we have $\lambda(\Delta_i) = \Delta_j$.

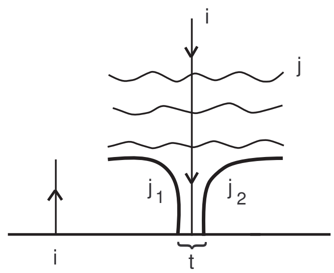


Figure 13: Splitting the arc j along the end of i

Now, to see that $\partial^{l(i)}$ is independent of the nonseparating type 1 arc i connecting ∂_k to itself, we prove the following claim:

Claim 1: If we start with two type 1 nonseparating arcs i and j starting and ending on ∂_k, then $\partial^{l(i)} = \partial^{l(j)}$.

Proof: Let i and j be nonseparating type 1 arcs connecting ∂_k to itself. W.L.O.G. we can assume that i and j have minimal intersection. First, we prove that there is a sequence $j = r_0 \to r_1 \to ... \to r_{n+1} = i$ of essential properly embedded nonseparating type 1 arcs joining ∂_k to itself so that each consecutive pair is disjoint, i.e. the isotopy classes of these arcs define a path in $\mathcal{B}(R)$, between i and j.

If $|i \cap j| = 0$, then take $r_0 = j$, $r_1 = i$. Assume that $|i \cap j| = m > 0$. We orient i and j arbitrarily. Then, we define two arcs in the following way: Start on the boundary component ∂_k, on one side of the beginning point of j and continue along j without intersecting j, till the last intersection point along i. Then we would like to follow i, without intersecting j, until we reach ∂_k. So, if we are on the correct side of j we do this; if not, we change our starting side from the beginning and follow the construction. This gives us an arc, say j_1. We define j_2, another arc, by changing the orientation of j and following the same construction. It is easy to see that j_1, j_2 are disjoint properly embedded arcs connecting ∂_k to itself as i and j do. One can see that j_1, j_2 are essential arcs since i, j intersect minimally. In Figure 13, we show the beginning and the end points of i, the essential intersections of i, j, and j_1, j_2

near the end point of i on ∂_k.

$|i \cap j_1| < m$, $|i \cap j_2| < m$ since we eliminated at least one intersection with i. We also have $|j_1 \cap j| = |j_2 \cap j| = 0$ since we never intersected j in the construction. Notice that j_1 and j_2 are not oriented, and i is oriented. It is easy to see from the construction that one of j_1 or j_2 has to be nonseparating type 1 arc, since j is a nonseparating type 1 arc.

Let $r_1 \in \{j_1, j_2\}$ and r_1 be nonseparating type 1. By the construction, we get $|i \cap r_1| < m$, $|j \cap r_1| = 0$. Now, using i and r_1 in place of i and j we can define a new nonseparating type 1 arc r_2, with the properties $|i \cap r_2| < |i \cap r_1|, |r_1 \cap r_2| = 0$. By an inductive argument, we get a sequence of nonseparating type 1 arcs such that every consecutive pair is disjoint, $i = r_{n+1} \to r_n \to r_{n-1} \to ... \to r_1 \to r_0 = j$. This gives us a special path in $\mathcal{B}(R)$ between i and j. By using Lemma 3.2 and Lemma 3.5, we can see a regular neighborhood of the union of each consecutive pair in the sequence and the boundary component of R that the arcs are starting and ending at, and encoding circles of these consecutive arcs. Then, by using the results of Lemma 3.6 and 3.7, we can see that each pair of disjoint nonseparating type 1 arcs give us the same boundary component. Hence, by using the sequence given above, we conclude that i and j give us the same boundary component. This proves Claim 1.

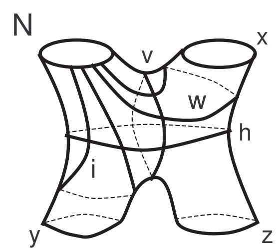


Figure 14: Unlinked arcs and their encoding circles

Let i_o be a properly embedded nonseparating type 1 arc on R connecting ∂_k to itself. Let $\partial^l = \partial^{l(i_o)}$. If i is a properly embedded, nonseparating type 1 arc on R connecting ∂_k to itself, then by the arguments given above we have $\partial^l = \partial^{l(i)}$, and there exists a properly embedded arc j on R connecting ∂^l to itself such that $\lambda(\Delta_i) = \Delta_j$.

Suppose that i is a nontrivial properly embedded separating type 1 arc on R connecting ∂_k to itself. Then it is easy to see that we can find an essential circle v and a nonseparating type 1 arc w, connecting ∂_k to itself such that $\partial_k \cup i \cup w$ has a regular neighborhood, N, which is a sphere with four boundary components, as shown in Figure 14. Let x, y, z, h, v be as shown in the figure. Notice that x and h are encoding circles of w. Since w is a nonseparating type 1

arc, x is essential. Then, by the proof of Lemma 3.6, there exist essential simple closed curves $x' \in \lambda([x]), h' \in \lambda([h]), v' \in \lambda([v]), N' \subseteq R$, $\partial^{k'}, y', z'$, where y', z' are boundary components of R if y, z are inessential circles respectively and $y' \in \lambda([y]), z' \in \lambda([z])$ is y, z are essential circles respectively and there exists a homeomorphism $\chi : (N', \partial^{k'}, x', y', z', h', v') \to_{\chi} (N, \partial_k, x, y, z, h, v)$.

Let i' be an arc connecting $\partial^{k'}$ to itself in the pair of pants determined by $\partial^{k'}, v', y'$ and w' be an arc connecting $\partial^{k'}$ to itself in the pair of pants determined by $\partial^{k'}, x', h'$. We have $\lambda(\Delta_i) = \Delta_{i'}$. Notice that since w is a nonseparating type 1 arc, so is w'. Since i' and w' connect $\partial^{k'}$ to itself, and w' is a nonseparating type 1 arc, by using the previous arguments, we see that $\partial^l = \partial^{k'}$. So, the correspondence that we get on boundary components of R using nonseparating type 1 arcs is the same as the one that we get by using separating type 1 arcs. Hence, ∂^l is the boundary component that we want. $\square$

We define a map $\sigma : \{\partial_1, ..., \partial_p\} \to \{\partial^1, ..., \partial^p\}$ using the correspondence which is given by Lemma 3.8.

Lemma 3.9 *Let $[i] \in \mathcal{V}(R)$. If i connects ∂_k to ∂_l on R where $k, l \in \{1, ..., p\}$, then there exists a unique $[j] \in \mathcal{V}(R)$ such that j connects $\sigma(\partial_k)$ to $\sigma(\partial_l)$ and $\lambda(\Delta_i) = \Delta_j$.*

Proof. Let $[i] \in \mathcal{V}(R)$ and let i connect ∂_k to ∂_l on R where $k, l \in \{1, ..., p\}$. If $\partial_k = \partial_l$, there exists a nontrivial properly embedded arc, j, connecting $\sigma(\partial_k)$ to itself such that $\lambda(\Delta_i) = \Delta_j$ by Lemma 3.8. If $\partial_k \neq \partial_l$, we can see the existence of j as follows: Let a be a properly embedded nontrivial nonseparating arc which connects ∂_k to itself and let b be a properly embedded nontrivial nonseparating arc which connects ∂_l to itself such that a, b, i are pairwise disjoint and they are on a subsurface, N, which is a sphere with four boundary components, as shown in Figure 15. Let h, v, y, z be as shown in the figure.

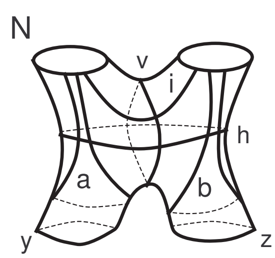


Figure 15: Arcs on sphere with four holes

Since a and b are nonseparating, y and z are essential circles. By Lemma 3.6, there exists a subsurface N', representatives h', v', y', z' in $\lambda([h]), \lambda([v]), \lambda([y]), \lambda([z])$ respectively and two boundary components, ∂^r, ∂^t of R and a homeomorphism $\chi : (N, \partial_k, \partial_l, v, h, y, z) \to (N', \partial^r, \partial^t, v', h', y', z')$. Then, by the proof of Lemma

3.8, we see that $\partial^r = \sigma(\partial_k), \partial^t = \sigma(\partial_l)$. Let j be a properly embedded arc connecting $\sigma(\partial_k)$ to $\sigma(\partial_l)$ in the pair of pants bounded by $\sigma(\partial_k), \sigma(\partial_l)$ and h' . Then, we have $\lambda(\Delta_i) = \Delta_j$.

Now, let e be an essential properly embedded arc in R such that e connects $\sigma(\partial_k)$ to $\sigma(\partial_l)$ and $\lambda(\Delta_i) = \Delta_e$. Then, we have $\Delta_e = \Delta_j = \lambda(\Delta_i)$. Let Q be a regular neighborhood of $e \cup \sigma(\partial_k) \cup \sigma(\partial_l)$. Since $\Delta_e = \Delta_j$, there is a properly embedded arc j_1 isotopic to j in Q. Then, since both j_1 and e connect the same boundary components in this pair of pants, they are isotopic. Then, $[j] = [e]$. Hence, $[j]$ is the unique isotopy class in R such that j connects $\sigma(\partial_k)$ to $\sigma(\partial_l)$ and $\lambda(\Delta_i) = \Delta_j$. $\square$

λ induces a unique map $\lambda_* : \mathcal{V}(R) \to \mathcal{V}(R)$ such that if $[i] \in \mathcal{V}(R)$ then $\lambda_*([i])$ is the unique isotopy class corresponding to $[i]$ where the correspondence is given by Lemma 3.9. Using the results of the following lemmas, we will prove that λ_* extends to an injective simplicial map on $\mathcal{B}(R)$.

Lemma 3.10 $\lambda_* : \mathcal{V}(R) \to \mathcal{V}(R)$ *extends to a simplicial map* $\lambda_* : \mathcal{B}(R) \to \mathcal{B}(R)$.

Proof. It is enough to prove that if two distinct isotopy classes of essential properly embedded arcs on R have disjoint representatives, then their images under λ_* have disjoint representatives. Let a, b be two disjoint representatives of two distinct classes in $\mathcal{V}(R)$. Let $\partial_1, ..., \partial_p$ be the boundary components of R. We consider the following cases:

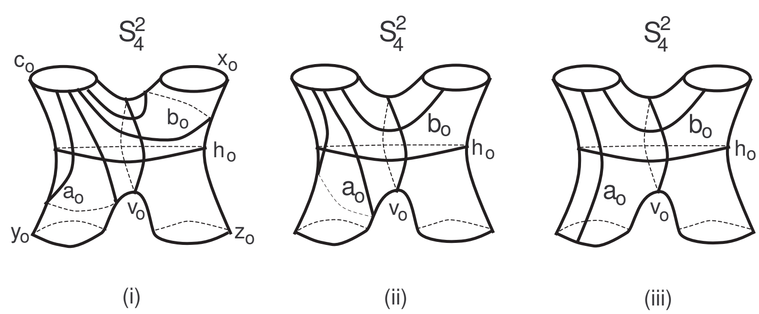


Figure 16: Arcs and their encoding circles

Case 1: Assume that there is no common boundary among the boundaries that a and b connect. Let ∂_i, ∂_j be the boundary components that a starts and ends, and let ∂_m, ∂_n be the boundary components that b starts and ends where $i, j = 1, ..., p$ and $m, n = 1, ..., p$ and $\{m, n\} \cap \{i, j\} = \emptyset$. Then, since $a \cup \partial_i \cup \partial_j$ is disjoint from $b \cup \partial_m \cup \partial_n$, we can find disjoint regular neighborhoods, N_1, of $a \cup \partial_i \cup \partial_j$ and N_2 of $b \cup \partial_m \cup \partial_n$ on R, which give us two disjoint pair of pants. Then, by using Lemma 2.12, and the definition of λ_*, it is easy to see that the corresponding arcs (images)

will have disjoint representatives.

Case 2: Assume that a, b are unlinked, connecting ∂_i to itself for some $i = 1, ..., p$. By Lemma 3.2, there is a homeomorphism ϕ such that $(S_4^2, a_o, b_o) \cong_\phi (N, a, b)$ where N is a regular neighborhood of $a \cup b \cup \partial_i$ in R and a_o, b_o are as shown in Figure 16 (i). Then, by using Lemma 3.6 and the definition of λ_*, we see that images have disjoint representatives.

Case 3: Assume that a connects one boundary component ∂_i to itself for some $i = 1, ..., p$, and b connects ∂_i to ∂_k for some $k \neq i$. Then, by Lemma 3.3, there is a homeomorphism ϕ such that $(S_4^2, a_o, b_o) \cong_\phi (N, a, b)$ where N is a regular neighborhood of $a \cup b \cup \partial_i \cup \partial_k$ in R and a_o, b_o are as shown in Figure 16 (ii). Then, by using Lemma 3.6 and the definition of λ_*, we see that the images have disjoint representatives.

Case 4: Assume that a connects ∂_i to ∂_j and b connects ∂_i to ∂_k where $k \neq i, i \neq j, j \neq k$. Then, by Lemma 3.4, there is a homeomorphism ϕ such that $(S_4^2, a_o, b_o) \cong_\phi (N, a, b)$ where N is a regular neighborhood of $a \cup b \cup \partial_i \cup \partial_k \cup \partial_j$ in R and a_o, b_o are as shown in Figure 16 (iii). Then, by using Lemma 3.6 and the definition of λ_*, as in the previous cases we see that the images have disjoint representatives.

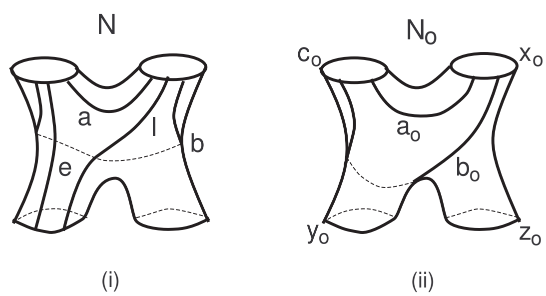


Figure 17: Arcs on sphere with four holes

Case 5: Assume that each of a and b connect two distinct boundary components, say ∂_i, ∂_j, of R.

We will first prove this case when R has at least 3 boundary components. Let ∂_k be a boundary component different from ∂_i and ∂_j. Let e and l be disjoint properly embedded arcs which are disjoint from a, b such that e connects ∂_i to ∂_k, l connects ∂_j to ∂_k and a, e, l are in a subsurface N, which is a sphere with 4 holes, of R as shown in Figure 17 (i). Then, by applying the result of case 4 to each pair in $\{a, e, l\}$, we can find disjoint representatives a_1, e_1 of $\lambda_*([a]), \lambda_*([e])$ respectively, disjoint representatives a_2, l_1 of $\lambda_*([a]), \lambda_*([l])$ respectively, disjoint representatives,

e_2, l_2 of $\lambda_*([e]), \lambda_*([l])$ respectively. Then, by using Lemma 3.4, and Lemma 3.6 we can choose disjoint representatives a', e', l' of $\lambda_*([a]), \lambda_*([e]), \lambda_*([l])$ respectively, a subsurface $N' \subseteq R$ and a homeomorphism $(N', a', e', l') \to (N, a, e, l)$ where N, a, e, l are as shown in Figure 17 (i). Since b and e are disjoint and b connects ∂_i to ∂_j and e connects ∂_i to ∂_k, by using case 4, we can choose a representative b_1 of $\lambda_*([b])$ which is disjoint from e'. Similarly, we can choose a representative b_2 of $\lambda_*([b])$ which is disjoint from l'. Then, since e' and l' are disjoint, we can choose a representative b_3 of $\lambda_*([b])$ which is disjoint from $e' \cup l'$. Then, it is easy to see that $\lambda([b])$ has a representative which is disjoint from a'.

Now, assume that R has exactly two boundary components, ∂_i, ∂_j. By Lemma 3.1, there is a homeomorphism $\phi : (N_o, a_o, b_o, c_o, x_o, y_o, z_o) \to (N, a, b, \partial_i, \partial_j, y, z)$ where N is a regular neighborhood of $a \cup b \cup \partial_i \cup \partial_j$ in R and $a_o, b_o, c_o, x_o, y_o, z_o$ are as shown in Figure 17 (ii). Since R has exactly two boundary components, y and z are essential circles in R. Then, using Lemma 2.12, we can choose pairwise disjoint representatives y', z' of $\lambda([y]), \lambda([z])$ respectively, boundary components ∂^k, ∂^l and a subsurface $N' \subseteq R$, and a homeomorphism $\chi : (N', \partial^k, \partial^l, y', z') \to (S^2_4, c_o, x_o, y_o, z_o)$.

N and N' are spheres with four holes in R. Since y, z are essential circles in R, the essential circles on N are essential in R. Similarly, since y', z' are essential circles in R, the essential circles on N' are essential in R. Furthermore, we can identify $\mathcal{C}(N)$ and $\mathcal{C}(N')$ with two subcomplexes of $\mathcal{C}(R)$ in such a way that the isotopy class of an essential circle in N or in N' is identified with the isotopy class of that circle in R. Now, suppose that α is a vertex in $\mathcal{C}(N)$. Then, with this identification, α is a vertex in $\mathcal{C}(R)$ and α has a representative in N. Then as in the proof of Lemma 3.6, we can see that λ maps $\mathcal{C}(N) \subseteq \mathcal{C}(R)$ to $\mathcal{C}(N') \subseteq \mathcal{C}(R)$. Then, since N and N' has four boundary components, we can apply the arguments given in the proof of the first part to see that there are disjoint representatives of $\lambda_*([a])$ and $\lambda_*([b])$ in N'. This proves case 5.

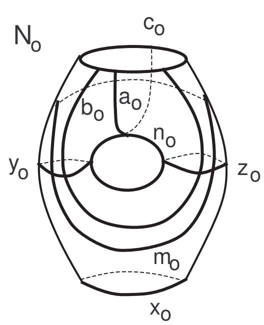


Figure 18: Linked arcs and their encoding circles

Case 6: Assume that a, b are linked, connecting ∂_i to itself for some $i = 1, ..., p$. By

Lemma 3.5, there is a homeomorphism $\phi : (N_o, a_o, b_o) \to (N, a, b)$ where N is a regular neighborhood of $\partial_i \cup a \cup b$ in R and N_o, a_o, b_o are as in Figure 18. Since y_o, z_o, c_o and m_o, n_o, c_o are the boundary components of regular neighborhoods of $a_o \cup c_o$ and $b_o \cup c_o$ on N_o respectively, $\phi(y_o), \phi(z_o)$ and $\phi(m_o), \phi(n_o)$ are encoding circles for a and b on R respectively. We have $(N, \phi(c_o), \phi(x_o), \phi(y_o), \phi(z_o), \phi(m_o), \phi(n_o)) \cong (N_o, c_o, x_o, y_o, z_o, m_o, n_o)$. Since the genus of R is at least 2, $\phi(x_o)$ is an essential circle on R. By Lemma 3.7, there exists ∂^j, $x'_o \in \lambda(\phi(x_o))$, $y'_o \in \lambda(\phi(y_o))$, $z'_o \in \lambda(\phi(z_o))$, $m'_o \in \lambda(\phi(m_o))$, $n'_o \in \lambda(\phi(n_o))$, $N' \subseteq R$ and a homeomorphism $\chi : (N_o, c_o, x_o, y_o, z_o, m_o, n_o) \to (N', \partial^j, x'_o, y'_o, z'_o, m'_o, n'_o)$. Since $\phi(y_o), \phi(z_o)$ and $\phi(m_o), \phi(n_o)$ are encoding circles for a and b on R respectively, y'_o, z'_o and m'_o, n'_o are encoding circles for $\lambda_*([a]), \lambda_*([b])$ on R respectively. Existence of χ shows that $\lambda_*([a]), \lambda_*([b])$ have disjoint representatives. $\chi(a_o)$ and $\chi(b_o)$ are disjoint representatives for $\lambda_*([a]), \lambda_*([b])$ respectively.

We have shown in all the cases that if two vertices have disjoint representatives, then λ_* sends them to two vertices which have disjoint representatives. Hence, λ_* extends to a simplicial map $\lambda_* : \mathcal{B}(R) \to \mathcal{B}(R)$. $\square$

Lemma 3.11 *Let $\lambda : \mathcal{C}(R) \to \mathcal{C}(R)$ be a superinjective simplicial map. Then, $\lambda_* : \mathcal{B}(R) \to \mathcal{B}(R)$ is injective.*

Proof. It is enough to prove that λ_* is injective on the vertex set, $\mathcal{V}(R)$. Let $[i], [j] \in \mathcal{V}(R)$ such that $\lambda_*([i]) = \lambda_*([j]) = [k]$. Then, by the definition of λ_*, the type of [i] and [j] are the same. Assume they are both type 1.1. Let $\{[x], [y]\}$ and $\{[z], [t]\}$ be the encoding simplices for $[i]$ and $[j]$ respectively. Then, $\{\lambda([x]), \lambda([y])\}$ and $\{\lambda([z]), \lambda([t])\}$ are encoding simplices of $[k]$. So, $\{\lambda([x]), \lambda([y])\} = \{\lambda([z]), \lambda([t])\}$. Then, since λ is injective, we get $\{[x], [y]\} = \{[z], [t]\}$. This implies $[i] = [j]$. The other cases can be proven similarly to the first case by using the injectivity of λ. $\square$

The following lemma can be proven similar to the proof of Lemma 4.13 in [3], which uses the Connectivity Theorem for Elementary Moves of Mosher, [9], appropriately restated for surfaces with boundaries. We will only state this lemma here.

Lemma 3.12 *If an injection $\mu : \mathcal{B}(R) \to \mathcal{B}(R)$ agrees with $h_* : \mathcal{B}(R) \to \mathcal{B}(R)$ on a top dimensional simplex, where h_* is induced by a homeomorphism $h : R \to R$, then μ agrees with h_* on $\mathcal{B}(R)$.*

Notation: A homeomorphism $g : R \to R$ induces a map $g_\# : \mathcal{C}(R) \to \mathcal{C}(R)$, where $g_\# = [g]$ and $g_\#$ induces a map $g_* : \mathcal{B}(R) \to \mathcal{B}(R)$ in a similar way as λ induces λ_*.

Remark: We have proven that λ is an injective simplicial map which preserves the geometric intersection 0 and 1. If the number of boundary components of R is at least 2, using these properties and following N.V.Ivanov's proof of his Theorem 1.1 [4], it can be seen that λ_* agrees with a map, $h_\#$, induced by a homeomorphism $h : R \to R$ on a top dimensional simplex in $\mathcal{B}(R)$. Then, by Lemma 3.12, it

agrees with $h_\#$ on $\mathcal{B}(R)$. Then, it is easy to see that λ agrees with a map, h_* on $\mathcal{C}(R)$.

If the number of boundary components of R is exactly 1, we cut R along a nonseparating simple closed curve, c on R. Let R_c be this cut surface and let c_+, c_- be the two boundary components of R_c which comes from cutting R along c. Then, considering how λ induced λ_* and using the techniques of [3], it is easy to see that λ induces a superinjective simplicial map $\lambda_c : \mathcal{C}(R_c) \to \mathcal{C}(R_d)$ and an injective simplicial map $(\lambda_c)_* : \mathcal{B}(R_c) \to \mathcal{B}(R_d)$, where $\lambda([c]) = [d]$. Then, since R_c and R_d have 3 boundary components, by adapting the arguments given in the paragraph above and using Lemma 2.12, we see that λ_c agrees with a map $(g_c)_\#$ on $\mathcal{C}(R_c)$, where $(g_c)_\#$ is induced by a homeomorphism $g_c : R_c \to R_d$ such that $g_c(\{c_+, c_-\}) = \{d_+, d_-\}$. We can do this argument for any nonseparating simple closed curve on R. Then, to see that λ agrees with a map, $h_\#$, which is induced by a homeomorphism $h : R \to R$, on $\mathcal{C}(R)$ we use the following lemma.

Lemma 3.13 *Let $\lambda : \mathcal{C}(R) \to \mathcal{C}(R)$ be a superinjective simplicial map. Assume that for any nonseparating simple closed curve c on R, λ_c agrees with a map, $(g_c)_\# : \mathcal{C}(R_c) \to \mathcal{C}(R_d)$, which is induced by a homeomorphism $g_c : R_c \to R_d$ where $g_c(\{c_+, c_-\}) = \{d_+, d_-\}$ and $\lambda([c]) = [d]$. Then, λ agrees with a map $h_\# : \mathcal{C}(R) \to \mathcal{C}(R)$ which is induced by a homeomorphism $h : R \to R$.*

Proof. Let c be a nonseparating simple closed curve and $(g_c)_\# : \mathcal{C}(R_c) \to \mathcal{C}(R_d)$ be a simplicial map induced by a homeomorphism $g_c : R_c \to R_d$ where $g_c(\{c_+, c_-\}) = \{d_+, d_-\}$ and $\lambda([c]) = [d]$ such that λ_c agrees with $(g_c)_\#$ on $\mathcal{C}(R_c)$. Let g be a homeomorphism of R which cuts to a homeomorphism $R_c \to R_d$ which is isotopic to g_c. Then each homeomorphism of R which cut to a homeomorphism $R_c \to R_d$ which is isotopic to g_c, is isotopic to an element in the set $\{gt_c^n, n \in \mathbb{Z}\}$, [1]. It is easy to see that λ_c agrees with $((gt_c^n)_c)_\#$ on $\mathcal{C}(R_c)$ for all $n \in \mathbb{Z}$.

Let w be a simple closed curve which is dual to c (i.e. w intersects c transversely once and there is no other intersection). Let P be a regular neighborhood of $c \cup w$. Then, P is a genus one surface with one boundary component. Let y be the boundary component of P. We have $i([c], [y]) = 0$, $i([w], [y]) = 0$, and $i([c], [w]) = 1$. Then, since λ is superinjective we have $i(\lambda([c]), \lambda([y])) = 0$, $i(\lambda([w]), \lambda([y])) = 0$ and $i(\lambda([c]), \lambda([w])) = 1$.

Let Q be the genus one subsurface with one boundary component of R which has $g(y)$ as its boundary. Then, it is easy to see that $g(c) = d \subseteq Q$, $g(w) \subseteq Q$ and $g(w)$ is dual to d, since w is dual to c.

Since $[y] \in \mathcal{C}(R_c)$, $\lambda([y]) = g_\#([y]) = [g(y)]$. We also have $[d] = \lambda([c]) = g_\#([c])$. Since $i(\lambda([c]), \lambda([y])) = 0$, $i(\lambda([w]), \lambda([y])) = 0$ and $i(\lambda([c]), \lambda([w])) = 1$, and $d \in \lambda([c])$, $g(y) \in \lambda([y])$ and d and $g(y)$ are disjoint, we can choose a simple closed curve $w' \in \lambda([w])$ such that w' is in $Int(Q)$ and dual to d. Then, $g^{-1}(w')$ is dual to c and $g^{-1}(w')$ is in $Int(P)$. Then, since both of w and $g^{-1}(w')$ are dual to c in $Int(P)$, there exists $m_c \in \mathbb{Z}$ such that $t_c^{m_c}([w]) = [g^{-1}(w')]$. Then, $gt_c^{m_c}([w]) = [w']$. Since

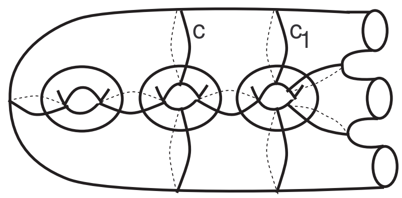


Figure 19: A configuration of circles

$\lambda([w]) = [w']$, $gt_c^{m_c}$ agrees with λ on $[w]$. We can identify $\mathcal{C}(R_c)$ with a subcomplex, L_c, of $\mathcal{C}(R)$. Let D_c be the set of isotopy classes of simple closed curves which are dual to c on R.

Claim 1: $(gt_c^{m_c})_\#$ agrees with λ on $\{[c]\} \cup L_c \cup D_c$.

Proof: It is clear that $(gt_c^{m_c})_\#([c]) = \lambda([c]) = [d]$. Since $(g_c)_\#$ agrees with λ_c on $\mathcal{C}(R_c)$, $(gt_c^{m_c})_\#$ agrees with λ on L_c.

We have seen that $gt_c^{m_c}$ agrees with λ on $[w]$. Let w_1 be a simple closed curve which is disjoint from w and dual to c. As we described before, there exists $\tilde{m_c} \in \mathbb{Z}$ such that λ agrees with $gt_c^{\tilde{m_c}}$ on $[w_1]$. Since w and w_1 are disjoint, $i(\lambda([w]), \lambda([w_1])) = 0$. If $m_c \neq \tilde{m_c}$ then $i((gt_c^{m_c})(w), gt_c^{\tilde{m_c}}(w_1)) \neq 0$ (since both w and w_1 are dual to c). Then, since $\lambda([w]) = (gt_c^{m_c})([w])$ and $\lambda([w_1]) = (gt_c^{\tilde{m_c}})([w_1])$, we would get $i(\lambda([w]), \lambda([w_1])) \neq 0$, which gives a contradiction. Therefore, $m_c = \tilde{m_c}$. Then, we see that $(gt_c^{m_c})_\#$ agrees with λ on $\{[c]\} \cup L_c \cup \{[w] \cup [w_1]\}$.

Given any curve t which is dual to c, by using similar techniques as in Lemma 3.8, we can find a sequence of dual curves to c, connecting d to t, such that each consecutive pair is disjoint, i.e. the isotopy classes of these curves define a path in $\mathcal{C}(R)$, between d and t. Then using the argument above and the sequence, we conclude that $(gt_c^{m_c})_\#$ agrees with λ on D_c. Hence, $(gt_c^{m_c})_\#$ agrees with λ on $\{[c]\} \cup L_c \cup D_c$. This proves claim 1. Let $h_c = gt_c^{m_c}$. We have that $(h_c)_\#$ agrees with λ on $\{[c]\} \cup L_c \cup D_c$.

Claim 2: Let v be a nonseparating simple closed curve on R. Then, $(h_c)_\# = (h_v)_\# = \lambda$ on $\mathcal{C}(R)$.

Proof: Since the genus of R is at least 2, we can find a sequence of nonseparating simple closed curves connecting c to v such that each consecutive pair is disjoint. Refining this sequence, we can get a sequence $c \to c_1 \to ... \to c_n = v$ of nonseparating simple closed curves connecting c to v such that each consecutive pair in this sequence is simultaneously nonseparating.

Let's consider the first consecutive pair in the sequence, c, c_1. Since $\{c, c_1\}$ is simultaneously nonseparating, it can be completed to a set G, (shown in Figure 19,

for $g = 3, p = 3$), see [6], such that the isotopy classes of Dehn twists about the elements of this set generate $PMod_R$ and all the curves in this set are (i) either disjoint from c or dual to c, and (ii) either disjoint from c_1 or dual to c_1. Then, since all the curves in G are either disjoint from c or dual to c, by Claim 1 we have that $(h_c)_\#([x]) = \lambda([x])$ for every $x \in G$. Similarly, since all the curves in G are either disjoint from c_1 or dual to c_1, by Claim 1 we have $(h_{c_1})_\#([x]) = \lambda([x])$ for every $x \in G$. Hence, $(h_c)_\#([x]) = \lambda([x]) = (h_{c_1})_\#([x])$ for every $x \in G$. Then, $(h_c)_\# = (h_{c_1})_\#$ since $(h_c^{-1} h_{c_1})_\# \in C(PMod_R) = \{1\}$.

By using our sequence, with an inductive argument we get that $(h_c)_\# = (h_v)_\#$ on $\mathcal{C}(R)$ and $(h_c)_\# = (h_v)_\# = \lambda$ on $\{[c]\} \cup \{[v]\} \cup L_c \cup D_c \cup L_v \cup D_v$. In particular we see that, $(h_c)_\#$ agrees with λ on any nonseparating curve v and on L_v. Since every separating curve is in the link, L_r, of some nonseparating curve r, we see that $(h_c)_\#$ agrees with λ on $\mathcal{C}(R)$. This proves the lemma. □

Proof of Theorem 1.1 follows from Lemma 3.13 and the remarks made before this lemma. Note that in [3], we referred to Ivanov's cutting arguments for the result of the corresponding theorem. The proof of the cutting argument given above can be adapted easily to that case.

4 Injective Homomorphisms of Subgroups of Mapping Class Groups

We assume that $\Gamma' = ker(\varphi)$ where $\varphi : Mod_R^* \to Aut(H_1(R, \mathbb{Z}_3))$ is the homomorphism defined by the action of homeomorphisms on the homology.

A mapping class $g \in Mod_R^*$ is called *pseudo-Anosov* if $\mathcal{A}$ is nonempty and if $g^n(\alpha) \neq \alpha$, for all α in $\mathcal{A}$ and any $n \neq 0$. g is called *reducible* if there is a nonempty subset $\mathcal{B} \subseteq \mathcal{A}$ such that a set of disjoint representatives can be chosen for $\mathcal{B}$ and $g(\mathcal{B}) = \mathcal{B}$. In this case, $\mathcal{B}$ is called a *reduction system* for g. Each element of $\mathcal{B}$ is called a *reduction class* for g. A reduction class, α, for g, is called an *essential reduction class* for g, if for each $\beta \in \mathcal{A}$ such that $i(\alpha, \beta) \neq 0$ and for each integer $m \neq 0$, $g^m(\beta) \neq \beta$. The set, $\mathcal{B}_g$, of all essential reduction classes for g is called the *canonical reduction system* for g. The correspondence $g \to \mathcal{B}_g$ is canonical. In particular, it satisfies $g(\mathcal{B}_h) = \mathcal{B}_{ghg^{-1}}$ for all g, h in Mod_R^*.

The following two lemmas are well known facts. The isotopy class of a Dehn twist about a circle a, is denoted by t_α, where $[a] = \alpha$.

Lemma 4.1 *Let $\alpha, \beta \in \mathcal{A}$ and i, j be nonzero integers. Then, $t_\alpha^i = t_\beta^j \Leftrightarrow \alpha = \beta$ and $i = j$.*

Lemma 4.2 *Let α, β be distinct elements in $\mathcal{A}$. Let i, j be two nonzero integers. Then, $t_\alpha^i t_\beta^j = t_\beta^j t_\alpha^i \Leftrightarrow i(\alpha, \beta) = 0$.*

The proofs of the following two lemmas follow by the techniques given in [3]. Note that we need to use that the maximal rank of an abelian subgroup of Mod_R^* is $3g-3+p$, [1], in these proofs.

Lemma 4.3 *Let K be a finite index subgroup of Mod_R^* and $f : K \to Mod_R^*$ be an injective homomorphism. Let $\alpha \in \mathcal{A}$. Then there exists $N \in \mathbb{Z}^*$ such that*

$$rank\ C(C_{\Gamma'}(f(t_\alpha^N))) = 1.$$

Lemma 4.4 *Let K be a finite index subgroup of Mod_R^*. Let $f : K \to Mod_R^*$ be an injective homomorphism. Then there exists $N \in \mathbb{Z}^*$ such that $f(t_\alpha^N)$ is a reducible element of infinite order for all $\alpha \in \mathcal{A}$.*

In the proof of Lemma 4.4, we use that centralizer of a p-Anosov element in the extended mapping class group is a virtually infinite cyclic group, [8].

Lemma 4.5 *Let K be a finite index subgroup of Mod_R^* and $f : K \to Mod_R^*$ be an injective homomorphism. Then $\forall \alpha \in \mathcal{A}$, $f(t_\alpha^N) = t_{\beta(\alpha)}^M$ for some $M, N \in \mathbb{Z}^*$, $\beta(\alpha) \in \mathcal{A}$.*

Proof. Let $\Gamma = f^{-1}(\Gamma') \cap \Gamma'$. Since Γ is a finite index subgroup we can choose $N \in Z^*$ such that $t_\alpha^N \in \Gamma$ for all α in $\mathcal{A}$. By Lemma 4.4 $f(t_\alpha^N)$ is a reducible element of infinite order in Mod_R^*. Let C be a realization of the canonical reduction system of $f(t_\alpha^N)$. Let c be the number of components of C and r be the number of p-Anosov components of $f(t_\alpha^N)$. Since $t_\alpha^N \in \Gamma, f(t_\alpha^N) \in \Gamma'$. By Theorem 5.9 [6], $C(C_{\Gamma'}(f(t_\alpha^N)))$ is a free abelian group of rank $c+r$. By Lemma 4.3 $c+r=1$. Then, either $c=1$, $r=0$ or $c=0$, $r=1$. Since there is at least one curve in the canonical reduction system we have $c=1$, $r=0$. Hence, since $f(t_\alpha^N) \in \Gamma'$, $f(t_\alpha^N) = t_{\beta(\alpha)}^M$ for some $M \in \mathbb{Z}^*$, $\beta(\alpha) \in \mathcal{A}$, [1], [5]. $\square$

Remark: Suppose that $f(t_\alpha^M) = t_\beta^P$ for some $\beta \in \mathcal{A}$ and $M, P \in \mathbb{Z}^*$ and $f(t_\alpha^N) = t_\gamma^Q$ for some $\gamma \in \mathcal{A}$ and $N, Q \in \mathbb{Z}^*$. Since $f(t_\alpha^{M \cdot N}) = f(t_\alpha^{N \cdot M})$, $t_\beta^{PN} = t_\gamma^{QM}$, $P, Q, M, N \in \mathbb{Z}^*$. Then, $\beta = \gamma$ by Lemma 4.1. Therefore, by Lemma 4.5, f gives a correspondence between isotopy classes of circles and f induces a map, $f_* : \mathcal{A} \to \mathcal{A}$, where $f_*(\alpha) = \beta(\alpha)$.

In the following lemma we use a well known fact that $f t_\alpha f^{-1} = t_{f(\alpha)}^{\epsilon(f)}$ for all α in $\mathcal{A}$, $f \in Mod_R^*$, where $\epsilon(f) = 1$ if f has an orientation preserving representative and $\epsilon(f) = -1$ if f has an orientation reversing representative.

Lemma 4.6 *Let K be a finite index subgroup of Mod_R^*. Let $f : K \to Mod_R^*$ be an injective homomorphism. Assume that there exists $N \in \mathbb{Z}^*$ such that $\forall \alpha \in \mathcal{A}$, $\exists Q \in \mathbb{Z}^*$ such that $f(t_\alpha^N) = t_\alpha^Q$. Then, f is the identity on K.*

Proof. We use Ivanov's trick to see that $f(k t_\alpha^N k^{-1}) = f(t_{k(\alpha)}^{\epsilon(k) \cdot N}) = t_{k(\alpha)}^{Q \cdot \epsilon(k)}$ and $f(k t_\alpha^N k^{-1}) = f(k) f(t_\alpha^N) f(k)^{-1} = f(k) t_\alpha^Q f(k)^{-1} = t_{f(k)(\alpha)}^{\epsilon(f(k)) \cdot Q}$ $\forall \alpha \in \mathcal{A}$, $\forall k \in K$. Then,

we have $t_{k(\alpha)}^{Q\cdot\epsilon(k)} = t_{f(k)(\alpha)}^{\epsilon(f(k))\cdot Q}$ $\forall\alpha \in \mathcal{A}$, $\forall k \in K$. Hence, $k(\alpha) = f(k)(\alpha)$ $\forall\alpha \in \mathcal{A}$, $\forall k \in K$ by Lemma 4.1. Then, $k^{-1}f(k)(\alpha) = \alpha$ $\forall\alpha \in \mathcal{A}$, $\forall k \in K$. Therefore, $k^{-1}f(k)$ commutes with t_α $\forall\alpha \in \mathcal{A}$, $\forall k \in K$. Since $PMod_R$ is generated by Dehn twists, $k^{-1}f(k) \in C(PMod_R)$ $\forall k \in K$. Since the genus of R is at least two and R is not a closed surface of genus two, $C(PMod_R)$ is trivial by 5.3 in [6]. So, $k = f(k)$ $\forall k \in K$. Hence, $f = id_K$. □

Corollary 4.7 *Let $g : Mod_R^* \to Mod_R^*$ be an isomorphism and $h : Mod_R^* \to Mod_R^*$ be an injective homomorphism. Assume that there exists $N \in \mathbb{Z}^*$ such that $\forall\alpha \in \mathcal{A}$, $\exists Q \in \mathbb{Z}^*$ such that $h(t_\alpha^N) = g(t_\alpha^Q)$. Then $g = h$.*

Proof. Apply Lemma 4.6 to $g^{-1}h$ with $K = Mod_R^*$. Since for all α in $\mathcal{A}$, $g^{-1}h(t_\alpha^N) = t_\alpha^Q$, we have $g^{-1}h = id_K$. Hence, $g = h$. □

By the remark after Lemma 4.5, we have that $f : K \to Mod_R^*$ induces a map $f_* : \mathcal{A} \to \mathcal{A}$, where K is a finite index subgroup of Mod_R^*. In the following lemma we prove that f_* is a superinjective simplicial map on $\mathcal{C}(R)$.

Lemma 4.8 *Let $f : K \to Mod_R^*$ be an injection. Let α, $\beta \in \mathcal{A}$. Then,*

$$i(\alpha, \beta) = 0 \Leftrightarrow i(f_*(\alpha), f_*(\beta)) = 0.$$

Proof. There exists $N \in \mathbb{Z}^*$ such that $t_\alpha^N \in K$ and $t_\beta^N \in K$. Then we have the following: $i(\alpha, \beta) = 0 \Leftrightarrow t_\alpha^N t_\beta^N = t_\beta^N t_\alpha^N$ (by Lemma 4.2) $\Leftrightarrow f(t_\alpha^N)f(t_\beta^N) = f(t_\beta^N)f(t_\alpha^N)$ (since f is injective on K) $\Leftrightarrow t_{f_*(\alpha)}^P t_{f_*(\beta)}^Q = t_{f_*(\beta)}^Q t_{f_*(\alpha)}^P$ where $P = M(\alpha, N), Q = M(\beta, N) \in \mathbb{Z}^*$ (by Lemma 4.5) $\Leftrightarrow i(f_*(\alpha), f_*(\beta)) = 0$ (by Lemma 4.2). □

Now, we prove the second main theorem of the paper.

Theorem 4.9 *Let f be an injective homomorphism, $f : K \to Mod_R^*$, then f is induced by a homeomorphism of the surface R and f has a unique extension to an automorphism of Mod_R^*.*

Proof. By Lemma 4.8 f_* is a superinjective simplicial map on $\mathcal{C}(R)$. Then, by Theorem 1.1, f_* is induced by a homeomorphism $h : R \to R$, i.e. $f_*(\alpha) = h_\#(\alpha)$ for all α in $\mathcal{A}$, where $h_\# = [h]$. Let $\chi^{h\#} : Mod_R^* \to Mod_R^*$ be the isomorphism defined by the rule $\chi^{h_\#}(k) = h_\# k {h_\#}^{-1}$ for all k in Mod_R^*. Then for all α in $\mathcal{A}$, we have the following:

$$\chi^{h_\#^{-1}} \circ f({t_\alpha}^N) = \chi^{h_\#^{-1}}(t_{f_*(\alpha)}^M) = \chi^{h_\#^{-1}}(t_{h_\#(\alpha)}^M) = h_\#^{-1} t_{h_\#(\alpha)}^M h_\# = t_{h_\#^{-1}(h_\#(\alpha))}^{M\cdot\epsilon(h_\#^{-1})} = t_\alpha^{M\cdot\epsilon(h_\#^{-1})}.$$

Then, since $\chi^{h_\#^{-1}} \circ f$ is injective, $\chi^{h_\#^{-1}} \circ f = id_K$ by Lemma 4.6. So, $\chi_\#^h|_K = f$. Hence, f is the restriction of an isomorphism which is conjugation by $h_\#$, (i.e. f is induced by h).

Suppose that there exists an automorphism $\tau : Mod_R^* \to Mod_R^*$ such that $\tau|_K = f$. Let $N \in Z^*$ such that $t_\alpha^N \in K$ for all α in $\mathcal{A}$. Since $\chi_\#^h|_K = f = \tau|_K$ and

$t_\alpha^N \in K$, $\tau(t_\alpha^N) = \chi_\#^h(t_\alpha^N)$ for all α in $\mathcal{A}$. Then, by Corollary 4.7, $\tau = \chi^{h_\#}$. Hence, the extension of f is unique. □

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